HIGHER ORDER SLIDING MODE CONTROL FOR CONTINUOUS TIME NONLINEAR SYSTEMS BASED ON OPTIMAL CONTROL

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Abstract: This paper addresses higher order sliding mode control for continuous nonlinear systems. We propose a new method of reaching control design while the sliding surface and equivalent control can be designed conventionally. The deviations of the sliding variable and its high order derivatives from zero are penalized. This is realized by minimizing the amplitudes of the higher order derivatives of the sliding variable. An illustrative example— a field-controlled DC motor— is given at the end.

1 INTRODUCTION

Variable structure systems (VSS) have been extensively used for control of dynamic industrial processes. The essence of variable structure control (VSC) is to use a high speed switching control scheme to drive the plant's state trajectory onto a specified and user chosen surface in the state space which is commonly called the sliding surface or switching surface, and then to keep the plant's state trajectory moving along this surface (Utkin, 1992), (Utkin, 1977). VSS has attracted attention during the past decades because of its superior capability to eliminate the impact of uncertainties.

Standard sliding mode controllers reveal drawbacks: high frequency vibration of the controlled system, which is also called "chattering", and sensitivity to disturbances during reaching mode. In recent years, a new method, so-called "higher order sliding mode (HOSM)" has been proposed (Levant, 1996), (Levant, 2007), (Glumineaus, 2006) for nonlinear sliding mode design. In higher order sliding mode problems, the switching controller also influences the higher order derivatives of the sliding variable rather than just the first order derivative. Under certain circumstances, for instance, the control u is treated as an additional state variable while its derivative $\dot{u}$ is employed as the actual control (Levant, 1996), (Zinober). The most popular higher order sliding mode controllers are the so called "twisting controller" and "super-twisting controller" which are derived based on bang-bang control theory. A number of papers report the derivation and performance of these controllers (Levant, 1996), (Levant, 2007), (Glumineaus, 2006), (Castellanos, 2004). As discussed by Boiko, Fridman and Castellanos (2004), if the actuator is of second or higher order there is an opportunity for reduction of the amplitude of chattering in the control signal when using twisting as a filter algorithm, compared with first order SM control. In other words, higher order sliding mode control contributes to suppressing the chattering effect although not completely eliminating it. Furthermore, a new concept, "integral sliding mode control (ISMC)" has been developed recently (Shi, 1996). With an integral sliding mode control scheme, the reaching phase is eliminated so that robustness is guaranteed right from the initial time instant.

The aim of this paper is to provide an effective and more convenient way to solve nonlinear higher order sliding mode problems. Nonlinear continuous systems are worked on and second or even higher order sliding mode control concepts are developed. With this method, a sliding mode is reached by forcing the sliding variable and its higher order derivatives to zero in finite time rather than working on nonlinear inequalities based on high order differential equations, which is inevitable in "super-twisting" controller design. The resulting reaching controller does not contain any high frequency switching component which evokes high frequency responses of the system. This idea is borrowed from optimal control laws. The derivation of equivalent control is different from that of normal sliding mode. Meanwhile, the

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sliding surface design may employ various methods.

In Section III B, we address the problem of the reachability of the sliding surface. To avoid chattering, whatever the initial state of the system is, both the sliding variable and its derivatives have to be driven to zero (not necessarily with the same convergence speed). They should also be kept at zero after the sliding surface is reached. In this paper, the reaching controller is expected to be a continuous nonlinear function with respect to the state variables. The form of the nonlinearity is determined by the solution of a minimization problem which is analogous to that which occurs in optimal control. What is to be minimized is the amplitude of the vector the entries of which are the sliding variable and its derivatives. If an q -th order sliding mode is pursued, the sliding variable and its derivatives up to order $(q-1)$ will be contained in the state vector. This method leads to a very smooth system trajectory. The reachablility of the sliding surface is guaranteed by the existence of a solution to the minimization operation. Furthermore, the minimization algorithm promises good robustness while the precision of high order sliding mode is kept.

At the end of this paper, a field controlled DC motor is considered. The performance of the proposed control scheme is shown applied to this third order system.

2 THE PROBLEM STATEMENT

Consider a continuous nonlinear system of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + Bu(t), t \geq t_0 && (1)\\ x(t_0) &= x_{0.} && (2) \end{aligned}$$

where $x(t) \in R^{n\times 1}$, $u \in R^{1\times 1}$ is the control input, σ is the sliding variable. $B^{n\times 1}, C, D$ are matrices or vectors of proper dimensions and n is known. It is assumed that $f(x(t))$ is Lipschitz continuous and differentiable with respect to $x(t)$ to any order. In this paper, the sliding variable is restrained to be a linear combination of the states, which has the following form:

$$\sigma(t) = Sx(t) = s_1x_1(t) + s_2x_2(t) + \ldots + s_nx_n(t) \quad (3)$$

Calculate the first and second order derivative of the sliding variable and we have

$$\dot{\sigma}(t) = S\dot{x}(t) = Sf(x(t)) + SBu(t) \quad (4)$$

$$\ddot{\sigma}(t) = S\ddot{x}(t) = S\frac{\partial f(x(t))}{\partial x(t)} f(x(t)) + S\frac{\partial f(x(t))}{\partial x(t)} Bu(t) + SB\dot{u}(t) \quad (5)$$

3 SECOND ORDER SLIDING MODE CONTROL DESIGN

3.1 Sliding Surface Design

For system (1), perform a similarity transformation defined by an orthogonal matrix $P^{n\times n}$:

$$x_l = Px = [x_{l1} : x_{l2}]^T,\ B_l = PB = \begin{bmatrix} 0_{k\times 1} \\ B_2 \end{bmatrix}, \quad (6)$$

$$f_l(x(t)) = f(x_l(t)) = \begin{bmatrix} f_{l1}(x(t)) \\ f_{l2}(x(t)) \end{bmatrix}. \quad (7)$$

where $x_{l1} \in R^{k\times 1}, x_{l2} \in R^{(n-k)\times 1}, B_2^{(n-k)\times 1}$ and x_{l1} does not have direct dependence on the input. Sliding surface design may be undertaken considering only x_{l1}, treating x_{l2} as an "input" to the partitioned system. In this way, the input may be ignored while determining the sliding surface and this reduces the complexity of the sliding surface design.

The partitioned state equations corresponding to (1) may now be expressed in the following way:

$$\begin{aligned} \dot{x_{l1}}(t) &= f_{l1}(x_{l1}(t), x_{l2}(t)) && (8)\\ \dot{x_{l2}}(t) &= f_{l2}(x_{l1}(t), x_{l2}(t)) + B_2u(t). && (9) \end{aligned}$$

Suppose

$$\begin{aligned} Sx(t) &= \begin{bmatrix} s_1 & s_2 & \cdots & s_n \end{bmatrix} x(t) \\ &= wPx(t) = wx_l(t) \\ &= w_{l1}x_{l1}(t) + w_{l2}x_{l2}(t) \end{aligned}$$

in which

$$\begin{aligned} w_{l1}^{p\times k} &= \begin{bmatrix} w_1 & w_2 & \cdots & w_k \end{bmatrix}, \\ w_{l2}^{p\times(n-k)} &= \begin{bmatrix} w_{k+1} & w_{k+2} & \cdots & w_n \end{bmatrix}, \end{aligned}$$

and $Sx(t)$ is the sliding variable, then the condition for the sliding mode to exist is

$$w_{l1}x_{l1}(t) + w_{l2}x_{l2}(t) = 0,$$

which yields

$$x_{l2}(t) = -w_{l2}^{-1} w_{l1} x_{l1}(t). \quad (10)$$

When w_{l2} is non-square, w_{l2}^{-1} in (10) should be its pseudo inverse.

Substituting (10) into (8) we have,

$$\begin{aligned} x_{l1}(t+1) &= f_{l1}(x_{l1}(t), -w_2^{-1}w_1x_{l1}(t)) && (11)\\ &= F(x_{l1}(t)) && (12) \end{aligned}$$

where $F(\cdot)$ denotes the nonlinear function about $x_{l1}(t)$ after tidying (11) up.

The goal of the next step is to fix the relationship between $x_{l2}(t)$ and $x_{l1}(t)$ to prescribe desirable performance for the nominal sliding mode dynamics. Any standard design algorithm which produces a linear state feedback controller for a nonlinear dynamic system can be used to determine $F(x_{l1}(t))$ and achieve desired performance through selection of sliding mode dynamics (Spurgeon, 1992). It is also assumed here that (12) is stabilizable. The controller gain derived is:

$$x_{l2}(t) = -kx_{l1}(t) \tag{13}$$

which means that

$$\sigma(\mathbf{x}_l(t)) = \begin{bmatrix} k & \vdots & I \end{bmatrix} x_l(t) \tag{14}$$

while I represents the identity matrix with proper dimension.

Note that inversion of the similarity transformation (using P) is needed to recover $x(t)$ from $x_l(t)$. Then $Sx(t) = 0$ is the desired sliding surface.

3.2 Higher Order Sliding Mode Design

3.2.1 Reaching Control Design

As the reaching condition implies, the sliding variable has to converge to zero in finite time. Furthermore, as an q-th order sliding mode is expected, $\dot{\sigma}$, $\ddot{\sigma}$......$\sigma^{(q-1)}$ are also desired to approach zero. Derive a vector containing $\sigma, \dot{\sigma}, \ddot{\sigma}$......$\sigma^{(q-1)}$ and extend (4), (5) to describe this vector

$$\dot{\sigma}(t) = S\dot{x}(t) = Sf(x(t)) + SBu(t) \tag{15}$$

$$\ddot{\sigma}(t) = S\ddot{x}(t) = S\frac{\partial f(x(t))}{\partial x(t)} f(x(t))$$
$$+S\frac{\partial f(x(t))}{\partial x(t)} Bu(t) + SB\dot{u}(t) \tag{16}$$

$$\sigma^{(3)}(t) = Sx^{(3)}(t)$$

$$= S(\frac{\partial^2 f(x(t))}{\partial x^{(2)}(t)} + (\frac{\partial f(x(t))}{\partial x(t)})^2) f(x(t)$$
$$+S(\frac{\partial^2 f(x(t))}{\partial x^{(2)}(t)} + (\frac{\partial f(x(t))}{\partial x(t)})^2) Bu(t)$$
$$+S\frac{\partial f(x(t))}{\partial(t)} B\dot{u}(t) + SBu^{(2)}(t), \tag{17}$$

......

which is equivalent to

$$z(t) = G(x(t)) + H(x(t))U(t). \tag{18}$$

where

$$z(t) = \begin{bmatrix} \sigma(t) \\ \dot{\sigma}(t) \\ \dots \\ \sigma^{(q-1)}(t) \end{bmatrix}, \ U(t) = \begin{bmatrix} u(t) \\ \dot{u}(t) \\ \dots \\ u^{(q-1)}(t) \end{bmatrix},$$

$$H(x(t)) =$$
$$\begin{bmatrix} 0 & 0 & \dots & 0 \\ SB & 0 & \dots & 0 \\ S\frac{\partial f(x(t))}{\partial x(t)}B & SB & \dots & 0 \\ S(\frac{\partial^2 f(x(t))}{\partial x^{(2)}(t)} + (\frac{\partial f(x(t))}{\partial x(t)})^2)B & \dots & \dots & \dots \\ \dots & \dots & \dots & SB \end{bmatrix},$$

$$G(x(t)) = \begin{bmatrix} Sx(t) \\ Sf(x(t)) \\ S\frac{\partial f(x(t))}{\partial x(t)} f(x(t)) \\ S(\frac{\partial^2 f(x(t))}{\partial x^{(2)}(t)} + (\frac{\partial f(x(t))}{\partial x(t)})^2) f(x(t)) \\ \dots \end{bmatrix} \tag{19}$$

Here, the following conditions are assumed:

Assumption I: $z(t) \in Z, Z$ *contains the origin.*

Assumption II: The set Z is reachable in finite time from any initial state and from any point in the generated trajectories.

As the purpose of reaching control design is to find some $u(t)$ which regulates $z(t)$ to zero in finite time, we define a cost function which is

$$J(t) = z^T(t)z(t) + \lambda U^T(t)U(t) \tag{20}$$

with a weighting factor λ. Then $U(t)$ is determined to minimize $J(t)$.

Taking the partial derivative of $J(t)$ with respect to $U(t)$ we have:

$$\frac{\partial J(t)}{\partial U(t)} = \frac{\partial(z^T(t)z(t) + \lambda U^T(t)U(t))}{\partial U(t)}. \tag{21}$$

Let $\frac{\partial J(t)}{\partial U(t)} = 0$ and derive:

$$U(t) = M(x(t))$$

$$= -(H(x(t))^T H(x(t)) + \lambda I)^{-1} H(x(t))^T G(x(t)) \tag{22}$$

(I here again represents the identity matrix if certain dimension.)

It should be noticed that the derivation of (22) reduces to a Tikhonov regularization problem therefore the detail is omitted here.

REMARK: Here, we assume the minimization over an infinite horizon results in a control $U^*(t)$. This control input will be implemented only until the next measurement becomes available. Then the up to date system information will be taken into account and a new value of $U^*(t)$ is computed. Introduce

$$\begin{aligned} J(t+h) &= z^T(t)z(t) + \lambda U^{*T}(t)U^*(t) && (23)\\ &\leq z^T(t)z(t) + \lambda U^T(t)U(t) = J(t) && (24) \end{aligned}$$

where $J(t)$ stands for the cost observed at time t and h is a sufficiently small positive number. The final cost $J(\infty)$ is a finite non-negative number as $J(t)$ is non-increasing. In other words, $J(t)$ decreases due to the effect of $U^*(t)$ until reaches zero. Then the next value (final value) of $U^*(t)$ is zero which indicates that the reaching mode is complete. Meanwhile, the final value of $z^T(t)z(t)$ is zero. By choosing λ to be a small positive weighting factor, non-zero $z^T(t)z(t)$ will be relatively heavily punished and so $z(t)$ converges to zero more quickly.

The reaching control law $u_r(t)$ can be obtained from the equation which forms the first row of (22) (Wertz, 1990)

$$u_r(t) = \begin{bmatrix} 1 & 0 & \ldots & \ldots & 0 \end{bmatrix} M(x(t)) = M_1(x(t)) \qquad (25)$$

where $M_1(x(t))$ stands for the first element of vector $M(x(t))$.

3.2.2 Robustness Issue

By substituting (22) into (18) we have

$$z(t) = G(x(t)) + H(x(t))M(x(t)). \qquad (26)$$

Now, assume that due to modelling errors, the real system is

$$\dot{x}(t) = f_{real}(x(t)) + B_{real}u(t), t \geq t_0 \qquad (27)$$

which leads to

$$z(t) = G_{real}(x(t)) + H_{real}(x(t))U(t). \qquad (28)$$

The robustness of the reaching mode relies on

- *Assumption I and II* for $z(t)$ in (28)
- The satisfaction of (29)

$$J(G_{real}, H_{real}, U^*(t), t) \leq J(G, H, t). \qquad (29)$$

3.2.3 Equivalent Control Design

After the sliding mode is reached, the system dynamic is dominated by the equivalent controller. To ensure $q-th$ order sliding, the equivalent control has to maintain $\sigma(t), \dot{\sigma}(t) \ldots \sigma^{(q-1)}(t)$ at zero. By extending (1), (3) we have

$$\sigma^{(q-1)}(t) = P(f(x)) + Q(u(t)). \qquad (30)$$

where $P(\cdot)$ and $Q(\cdot)$ are both nonlinear functions.

The equivalent control $u_{eq}(t)$ should be derived according to the following

$$\sigma^{(q-1)}(t) = P(f(x)) + Q(u_{eq}(t)) = 0. \qquad (31)$$

As introduced in (Matthews, 1988), the complete sliding mode controller is

$$u(t) = u_{eq}(t) + u_r(t) \qquad (32)$$

where $u_r(t)$ is from (25).

4 EXAMPLE AND SIMULATION RESULTS

4.1 Field controlled DC Motor and controller Design

Consider the example of a field-controlled DC motor. DC motors are widely used by almost all industries and can be highly nonlinear in field controlled configurations. The mathematical model of a DC motor can

Figure 1: Structure of a DC motor.

be expressed in the following way.

$$\begin{aligned} \dot{x_1}(t) &= -ax_1(t) + u(t) && (33)\\ \dot{x_2}(t) &= -bx_2(t) + \rho - cx_1(t)x_3(t) && (34)\\ \dot{x_3}(t) &= \theta x_1(t)x_2(t) && (35) \end{aligned}$$

$$y(t) = x_3(t). \qquad (36)$$

The physical meanings of the variables in the above equations are:

$x_1(t)$	Field current
$x_2(t)$	Armature current
$x_3(t)$	Angular velocity ,
$u(t)$	Field voltage
ρ	Armature voltage,

with a, b, c, θ, ρ positive constants.

The equilibria of the system are

$$x_1 = 0, x_2 = \frac{\rho}{b} \text{ and } x_3 = \omega_0,$$

where ω_0 is a desired setpoint for the angular velocity.

In this paper, we choose

$$a = b = c = \theta = \rho = 1$$

for simplicity (**?**).

The partitioned system matrices are

$$f_{l1}(x_{l1}, x_{l1}) = -x_1(t) \quad (37)$$

$$f_{l2}(\dot{x_{l1}}, x_{l1}) = \begin{bmatrix} -x_2(t) + 1 - x_1(t)x_3(t) \\ x_1(t)x_2(t) \end{bmatrix} \quad (38)$$

$$B_l = \begin{bmatrix} B_1 \\ 0_{2\times 1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (39)$$

It is seen that (33)-(35) are already in the same form as (6)-(7). Hence the transformation matrix P is identity.

Suppose

$$Sx(t) = [\ s_1 \quad s_2 \quad s_3\ ]x(t) = w_{l1}x_{l1}(t) + w_{l2}x_{l2}(t),$$

The values of w_1 and w_2 must be chosen to ensure the following system has satisfactory closed loop behavior:

$$x_{l2}(t) = Kx_{l1}(t) = -w_{l2}^{-1}w_{l1}x_{l1}(t)$$

$$\dot{x_{l1}}(t) = f_{l1}(x_{l1}(t), -w_{l2}^{-1}w_{l1}x_{l1}(t)) \quad (40)$$

$$= Fx_{l1}(t). \quad (41)$$

One of the proper selections of w_1 and w_2 leads to:

$$K = [\ 0 \quad -1\ ]$$

which produces a sliding variable:

$$\sigma(t) = Sx(t) = x_1(t) + x_3(t) - x_{3desired}$$

In this case, $x_3(t) - x_{3desired}$ is treated as the state of the system rather than $x_3(t)$ in certain design steps because the final value of x_3 is not expected to be zero but a desired value. This desired value should be involved in the sliding surface design. Similarly, $x_{2desired}$ should be considered at some stage as well. (In this case, $x_{2desired} = 0.95$ and $x_{3desired} = 2.05$.) Accordingly, we have

$$u_{eq}(t) = \frac{2x_1^2(t)x_3(t) + 2x_1(t)x_3(t) - x_1^3(t)x_2(t) - 4x_1(t)}{3x_2(t) + x_1(t)x_3(t) + 2x_3(t) - 3}$$

which is derived by letting

$$\sigma^{(3)}(t) = Sx^{(3)}(t) = 0.$$

Now we proceed to design the reaching control. In this case, take the derivatives of $\sigma(t)$ up to order 3 into account in the cost function definition. Then $G(x(t))$ and $H(x(t))$ can be calculated from (19). As the result of the minimization, the reaching control will be expressed in the form of a nonlinear function of state variables:

$$u_r(t) = W(x(t)).$$

$W(x(t)$ is derived from (22) and (25). Computing $W(x(t))$ is reduced to a numerical calculation without necessity of pursuing the algebraic description of $W(x(t))$. Finally the complete control law $u(t)$ is derived using (32) with the equivalent control derived according to (31).

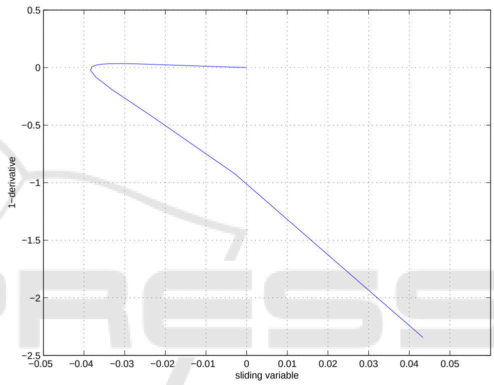

Figure 2: Sliding variable and its derivative.

4.2 Simulation Results

The integration step size is chosen to be $1ms$. In all the figures below, the unit of time axis is in second. The process depicting the sliding variable and its derivative as they approach zero is shown in Fig. 2 The trajectory travels smoothly on the plane until it reaches the origin without overshooting. From Fig. 3

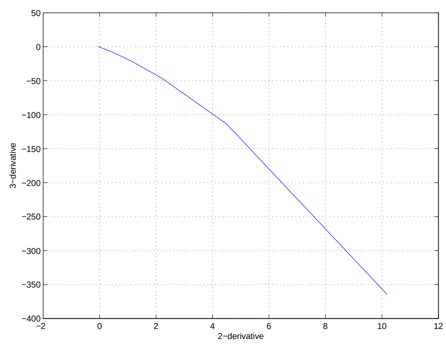

Figure 3: Higher order derivatives of the sliding variable.

we can see that the second and third order derivatives of the sliding variable also behave as a smooth curve which ends up at the origin. Figure 4 shows the convergence performance of the state variables. It is shown that $x_1(t)$ converges to zero while $x_2(t)$ and $x_3(t)$ each approach their desired value. The trajectories are smooth and there is no overshoot or oscillation. The whole process settles quickly within 0.5 seconds.

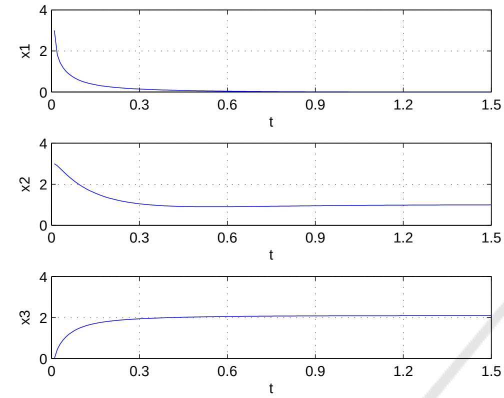

Figure 4: Convergence of the states.

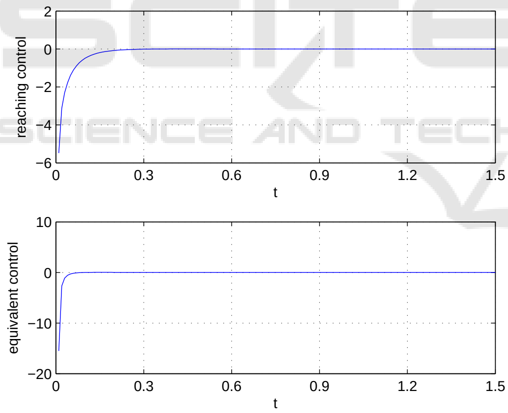

Figure 5: Control signal u.

The variation of the control signal u during the period is plotted in Figure 5.

As shown above, a good performance is achieved. A higher order sliding behavior is shown.

5 CONCLUSIONS

In this paper, a new method of designing a higher order sliding mode controller for a continuous nonlinear dynamic system is reported. Retaining the advantages of higher order sliding mode control, i.e. chattering reduction, the complexity of nonlinear design is greatly reduced with this method especially in reaching control design. A field-controlled DC motor is given as an illustrative example to show the effectiveness of this method.

REFERENCES

Vadim I. Utkin (1992). *Sliding Modes in Control and Optimization*. Springer-Verlag New York, Inc.

Vadim I. Utkin (1977). *Variable Structure Systems with Sliding Modes*. IEEE Transactions on Automatic Control, Volume AC-22, No. 2, April.

S. V. Emeryanov, S. K. Korovin, and A. Levant (1996). *Higher-order Sliding Modes in Control Systems*. Computational Mathematics and Modeling, Vol. 7, No. 3.

A. Levant (2007). *Principles of 2-sliding mode design*. Automatica Vol. 43, pp. 576 – 586.

S. Laghrouche, F. Plestan, and A. Glumineau (2005). *Multivariable practical higher order sliding mode control*. Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference.

A.J. Koshkouei, K.J. Burnham and A.S.I. Zinober. *Dynamic sliding mode control design*. IEE Proceedings online no. 20055133.

I. Boiko, L. Fridman, and I. M. Castellanos (2004). *Analysis of second-order sliding-mode algorithms in the frequency domain*. IEEE Transaction on Automatic Control, Vol. 49, No. 6, pp. 946–950, Jun.

Vadim Utkin and Jingxin Shi(1996). *Integral Sliding Mode in Systems Operating under Uncertainty Conditions*. Proceedings of the 35th Conference on Decision and Control, Kobe, Japan, December.

S. K. Spurgeon (2004). *Temperature Control of Industrial Process using a Variable Structure Design Philosophy*. Transactions of the Institute of Measurement and Control

R. R. Bitmead, M. Gevers, and V. Wertz (1990). *Adaptive Optimal Control: The Thinking Man's GPC*. Englewood Cliffs, NJ: Prentice-Hall.

Raymond, A. DeCarlo, Stanislaw H. Zak, Gregory P. Matthews(1988). *Variable Structure Control of Nonlinear Multivariable Systems: A Tutorial*. Proceedings of The IEEE, Vol. 76, No. 3, March.

Christopher Edwards and Sarah K. Spurgeon (1998). *Sliding Mode Control, Theory And Applications*. CRC Press, Taylor & Francis Group.