ON THE JOINT ESTIMATION OF UNKNOWN PARAMETERS AND DISTURBANCES IN LINEAR STOCHASTIC TIME-VARIANT SYSTEMS

Stefano Perabò and Qinghua Zhang

IRISA, INRIA Rennes, Campus universitaire de Beaulieu, Avenue du General Leclerc, 35042 Rennes Cedex, France

Abstract: Motivated by fault detection and isolation problems, we present an approach to the design of unknown parameters and disturbances estimators for linear time-variant stochastic systems. The main features of the proposed method are: (a) the joint estimation of parameters and disturbances can be carried out; (b) it is a full-stochastic approach: the unknown parameters and disturbances are random quantities and prior information, in terms of means and covariances, can be easily taken into account; (c) the estimator structure is not fixed a priori, rather derived from the optimal infinite dimensional one by means of a sliding window approximation. The advantages with respect to the widely used parity space approach are presented.

1 INTRODUCTION

The following discrete time linear stochastic system is considered in this brief paper:

\begin{align}
    x_{k+1} &= A_k x_k + B_k u_k + \Psi_k p + E_k d_k + w_k \\
    y_k &= C_k x_k + v_k
\end{align}

for \( k \geq 0 \), with \( A_k \in \mathbb{R}^{n \times n} \), \( B_k \in \mathbb{R}^{n \times m} \), \( \Psi_k \in \mathbb{R}^{n \times q} \), \( E_k \in \mathbb{R}^{p \times n} \) and \( C_k \in \mathbb{R}^{l \times n} \) known time-variant matrices. The vector sequences \( \{ x_k \} \), \( \{ u_k \} \) and \( \{ y_k \} \) denote respectively the state, input and output stochastic processes. The sequences \( \{ w_k \} \) and \( \{ v_k \} \) are assumed to be zero mean, white and uncorrelated wide-sense stochastic processes, with \( \mathbb{E}[w_k w_k^T] = Q_k \) and \( \mathbb{E}[v_k v_k^T] = R_k \) (positive definite), where \( \mathbb{E}[\cdot] \) denotes the mathematical expectation operator. The initial condition \( x_0 \) has known mean \( \mathbb{E}[x_0] = \mu_0 \) and covariance \( \mathbb{E}[(x_0 - \mu_0) (x_0 - \mu_0)^T] = P_0 \). Both the initial condition \( x_0 \) and the input process \( \{ u_k \} \) are assumed uncorrelated with the noise sequences.

The term \( E_k d_k \) accounts for unknown disturbances acting on the system or faults, whence the sequence \( \{ d_k \} \) is an unknown (and uncontrolled) input modeled as a wide-sense stochastic process, not necessarily stationary. The disturbances are further assumed uncorrelated with the initial state, the noise and the input processes, respectively \( x_0 \), \( \{ w_k \} \), \( \{ v_k \} \) and \( \{ u_k \} \).

Finally, the term \( \Psi_k p \) can account for the occurrence of parametric faults in the system (for instance with the meaning that when \( p \) is zero no faults are present) or for constant parameters that need to be estimated on-line. Here \( p \) is a random variable uncorrelated with the noise, input and disturbance processes.

The problem to be solved is the following: find for each \( N \geq 0 \) the minimum variance unbiased linear estimators of the disturbances sequence \( d_k^{N-1} = \{ d_k : 0 \leq k \leq N-1 \} \) and of the parameters \( p \), given the input and output sequences \( u_k^{N-1} \) and \( y_k^N \) and the conditions guaranteeing the uniqueness of the corresponding estimates. These estimators will be denoted respectively by \( \hat{d}_{k|N} \) and \( \hat{p}_{|N} \) (since \( p \) does not depend on time).

The following two related problems will also be discussed in this paper. First, how to weaken the uniqueness conditions by considering the quantities \( \hat{d}_{k|N} \) for \( 0 \leq k \leq N-1 \) and some appropriate delay \( D > 0 \), which will be called, with an abuse of terms, “delayed estimators”. Second, how to recursively and reliably compute the estimates \( \hat{p}_{|N+D} \) and \( \hat{d}_{k|N+D} \) once sample paths (measurements) \( u_k^{N+D-1} \) and \( y_k^{N+D} \) of the input and output processes are becoming available (by convention, italic characters will denote samples from the corresponding random variables which, in-
2 BASIC EQUATIONS FOR ESTIMATION

Pretend for a while that the parameters and the disturbances sequence are known quantities, i.e. as if they were inputs of the system described by (1), and assume the following:

**Assumption 1.** $\{A_k, C_k\}$ is uniformly completely observable and $\{A_k, C_k^{1/2}\}$ is uniformly completely reachable.

**Assumption 2.** The parameters $p$ and the disturbance sequence $\{d_k\}$ are uncorrelated from the initial state $x_0$ and the noise sequences $\{w_k\}$ and $\{v_k\}$.

Hence there is no feedback from the output to the parameters and disturbances (see Gevers and Anderson, 1982) for details) and by applying well known results of the linear estimation theory (Kailath et al., 2000), the following innovation representation of the output process $\{y_k\}$ can be derived:

$$
\hat{x}^e_{k+1} = A_k \hat{x}^e_{k|k-1} + B_k u_k + \Psi_k p + E_k d_k + K e^*_k
$$

$$
y_k = C_k \hat{x}^e_{k|k-1} + e^*_k, \tag{2a}
$$

the recursion being initiated setting $\hat{x}^e_0 = x_0$, where $\hat{x}^e_{k|k}$ is the one step minimum variance unbiased linear predictor of the state. Each term of the innovation sequence $\{e^*_k\}$ has zero mean and covariance $\Lambda_k$ given by the recursive solution of the same Riccati equation which is solved in the standard Kalman filter (i.e. with no disturbances and unknown parameters).

With respect to this one, however, the superscript * in (2) emphasizes that the "estimates" $\{\hat{x}^e_{k+1|k}\}$ cannot be computed because the realizations $p$ and $\{d_k\}$, of $p$ and $\{d_k\}$ respectively, are not really available. Also the gains $K_k$ are computed exactly as in the Kalman filter. By defining recursively the quantities,

$$
Y_0 = O \quad Y_{k+1} = (A_k - K_k C_k) Y_k + \Psi_k
$$

$$
s_0 = 0 \quad s_{k+1} = (A_k - K_k C_k) s_k + E_k d_k
$$

$$
z_0 = x_0 \quad z_{k+1} = (A_k - K_k C_k) z_k + B_k u_k + K_k y_k \tag{3c}
$$

and by using (2b) it is not difficult to check that the following is true:

$$
C_k s_k + C_k Y_k p + e^*_k = y_k - C_k z_k. \tag{4}
$$

Note that a realization of the sequence $\{z_k\}$ can be computed from available data only, i.e. system matrices, input and output sequences. As a matter of fact, (3c) is exactly the Kalman filter equation that would be obtained if $p \equiv 0$ and $d_k \equiv 0$ for all $k$.

It is possible to arrange in matrix form the set of equations obtained from (4) when $k = 1, 2, \ldots, N$. For example for $N = 4$ one obtains

$$
\begin{bmatrix}
C_0 \Phi_0 E_0 & C_0 \Phi_0 E_1 & C_0 \Phi_0 E_2 & C_0 \Phi_0 E_3 & C_0 Y_1 \\
C_1 \Phi_0 E_0 & C_1 \Phi_0 E_1 & C_1 \Phi_0 E_2 & C_1 \Phi_0 E_3 & C_1 Y_1 \\
C_2 \Phi_0 E_0 & C_2 \Phi_0 E_1 & C_2 \Phi_0 E_2 & C_2 \Phi_0 E_3 & C_2 Y_1 \\
C_3 \Phi_0 E_0 & C_3 \Phi_0 E_1 & C_3 \Phi_0 E_2 & C_3 \Phi_0 E_3 & C_3 Y_1
\end{bmatrix}
\begin{bmatrix}
d_1 \\
d_2 \\
d_3 \\
d_4
\end{bmatrix}
\begin{bmatrix}
\vec{e}^*_1 \\
\vec{e}^*_2 \\
\vec{e}^*_3
\end{bmatrix}
= \begin{bmatrix}
y_2 - C_2 z_2 \\
y_3 - C_3 z_3 \\
y_4 - C_4 z_4
\end{bmatrix}, \tag{5}
$$

where the transition matrices $\Phi^t$ are defined by

$$
\Phi^t_k = I, \quad \Phi^t_k = (A_k - K_k C_k) \Phi^t_{k-1}. \tag{6}
$$

For an arbitrary $N$, left multiply the above system by the block diagonal matrix $\mathbf{B} \mathbf{D} \mathbf{A}$ in such a way that the covariance of the zero mean vector $e^* = \text{vec}[\Lambda_1^{-1/2} e_N \ldots \Lambda_1^{-1/2} e_1]$ is equal to the identity matrix. A system of the form

$$
A_g + e^* = r
$$

is thus obtained, where the matrix $A \in \mathbb{R}^{N \times (N + N \times q)}$ has the same structure as in (5), $g = \text{vec}[d_0 \ldots d_{N-1} p]$ is the unknown term, and the vector $r = \text{vec}[r_N \ldots r_1]$ contains the computable residuals

$$
r_k = \Lambda_k^{-1/2}(y_k - C_k z_k). \tag{7}
$$

If $d_k \equiv 0$ for each $k$ and $p \equiv 0$, then $r = e^*$, i.e. the vector of residuals has zero mean and its covariance equals the identity matrix. Any statistical test indicating a deviation from this condition can be used to detect the presence of non-null disturbances and/or parameters.

Since samples of $r$ are available but instead $e$ cannot be observed, the most appealing approach to estimate $g$ is to compute its minimum variance linear estimator $\hat{g}$ given the random vector $r$. Thanks to the Assumption 2, the following holds:

$$
\mathbb{E}[e^*_k d^*_h] = O \quad \mathbb{E}[e^*_k p^T] = O \quad \forall k, h \geq 0. \tag{9}
$$
As a result, $g$ and $e^*$ in (7) are in fact uncorrelated. Provided that prior information on the random vector $g$ is given in terms of its mean $\bar{g}$ and covariance $\Sigma_g$ (assume $\Sigma_g$ invertible and the factorization $\Sigma_g^{-1} = B^TB$), a straightforward application of linear estimation formulas shows that $\hat{g}$ and the covariance of the error $\tilde{g} = g - \hat{g}$ can be obtained from

$$(A^TA + B^TB) (\tilde{g} - \mu_g) = A^T(r - \lambda u_k) \quad (10a)$$

$$\Sigma_{\tilde{g}} = (A^TA + B^TB)^{-1} \quad (10b)$$

One could suspect, at this point, that the information about the unknown terms which is available from knowledge of the input and output sequences, is not fully exploited if the only quantities that are used for the estimation of the disturbances and parameters are the residuals defined in (8). However, as long as linear estimators are considered, it is possible to prove that the method is optimal in the sense that, by estimating $g$ from (7) (instead of a different linear relation with the measurable sequences $(u_{kn}^{N-1}, y_{kn}^N)$) one in fact minimizes the estimation error variance.

When sample paths of the input and output sequences, say $(u_{0}^{N-1})$ and $(y_{0}^N)$, are available, one is faced to the problem of computing numerically the estimate $\hat{g} = \text{vec}[\hat{d}_{0N} \ldots \hat{d}_{N-1|N}]$ from the vector $\hat{r}$ denoting the realization of $r$. To this end, the availability or lack of prior information makes a difference. In the following the latter case is discussed.

3 NO PRIOR INFORMATION

3.1 Estimability Conditions

The absence of prior information about $g$ can be dealt with by setting $\mu_g = 0$ and letting $\Sigma_g \to \infty$ (or equivalently $\Sigma_g^{-1} \to 0$) which corresponds to a very large uncertainty. Formula (10a) becomes $(A^TA) \hat{g} = A^T \hat{r}$ which is the system of normal equations for computing the unique least squares solution of

$$Ag = r. \quad (11)$$

in the unknown $g$, provided that the matrix $A$ has full column rank. From a practical point of view, it should be noted that the proposed method requires simply checking the rank of matrices and solving least squares problems, for which efficient numerical tools are readily available. But, unfortunately, finding general estimability condition in analytic form, is a very complex task. The following is not difficult to prove:

**Proposition 1.** For a given $N \geq 1$, the estimates $\hat{p}_{|N}$ and $\hat{d}_{|N}$ for $0 \leq k \leq N - 1$ are unique if and only if the matrix $A$ in (11) has full column rank. Moreover, the uniqueness holds only if the following necessary conditions are satisfied:

$$(C1) \quad \text{rank} \begin{bmatrix} E_k & 0 & \ldots & 0 & \Psi_0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & E_k & \ldots & 0 & \Psi_1 \end{bmatrix} = rN + q \quad (12a)$$

$$(C2) \quad \text{rank} \left( \sum_{k=1}^{N} Y_k^T C_k^T C_k \right) = q. \quad (12b)$$

If rank($E_k$) = $f$ for all $k \geq 0$ and (C1) is true for a value $N = N_{\text{min}}$, then it is satisfied for all values $N \geq N_{\text{min}}$. Analogously, if (C2) is true for a value $N = N_{\text{min}}$, then it is satisfied for all values $N \geq N_{\text{min}}$.

3.2 Delayed Estimation

Consider first the case when there are no unknown parameters ($q = 0$). A sufficient (but not necessary) condition to ensure that $A$ has full column rank for all $N \geq 1$, hence the uniqueness of the estimates $\hat{d}_{|N}$ for $0 \leq k \leq N - 1$, is the following:

$$(C3) \quad \text{rank}(C_k E_k) = f \quad \forall k \geq 0. \quad (13)$$

However, when (C3) is not satisfied, it could still be possible to compute, for some delay $D > 0$, unique delayed estimates $\hat{d}_{|N-D}$ for $0 \leq k \leq N - 1$. To exemplify what has been just asserted, consider the case $C_k E_k = 0$ and thus (C3) is not satisfied (this situation may happen typically when $C_k$ and $E_k$ have both some zero entries, for example $C_k = [1 \ 0]$ and $E_k = [0 \ 1]^T$). Then the zero blocks appear in the term $Ag$ in (7) as shown in the following scheme (suppose, for example, that $N = 4$):

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td></td>
<td></td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>*</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>*</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

It is evident that $\hat{d}_{|N}$ (in the example above) is not estimable from measurements collected till time $N$ (in other words $\hat{d}_{|N}$ is not unique). However, if the blocks marked with a +, i.e. the matrices $C_{k+2} \Phi_{k+2} E_k$ in (7), have full column rank, it is sufficient to add the measurements at time $N + 1$ (at time 5 to continue the example) so that the unique estimates $\hat{d}_{|N+1}$ for $k = 0, \ldots, N - 1$ and, in particular $\hat{d}_{|N-1}$ in (in the example $\hat{d}_{|3}$), could be computed. The above argument can be generalized as follows: if for some $D > 0$ the conditions

$$(C4a) \quad \text{rank}(C_{k+D} \Phi_{k+D} E_k) = f \quad \forall k \geq 0 \quad (14a)$$

$$(C4b) \quad \begin{bmatrix} C_{k+D} \Phi_{k+D} E_k \\ C_{k+2} \Phi_{k+2} E_k \\ C_{k+1} E_k \end{bmatrix} = O \quad (14b)$$
are satisfied, then the estimates $\hat{d}_{kN+D}$ for $0 \leq k \leq N-1$ are unique (even if $A$ has not full rank).

When there are unknown parameters ($q > 0$), the conditions in (13) or (14) are no longer sufficient and, in general, the rank of the matrix $A$ has to be checked numerically. However note the following result:

**Proposition 2.** (a) Assuming that condition (C3) in (13) is satisfied, if the estimates $\hat{p}_N$ and $\hat{d}_{kN}$ for $0 \leq k \leq N-1$ are unique (i.e. the matrix $A$ has full column rank) for a value $N = N_{\text{min}}$, then they are unique also for all $N \geq N_{\text{min}}$.
(b) Analogously, assuming that conditions (C4) in (14) are satisfied, if the delayed estimates $\hat{p}_{N+D}$ and $\hat{d}_{kN+D}$ for $0 \leq k \leq N-1$ are unique for a value $N = N_{\text{min}}$, then they are unique also for all $N \geq N_{\text{min}}$.

### 3.3 Approximate Recursive Estimation

In order to compute the estimates from (11), a grow-

### 3.4 Comparison with the Parity Space Approach

In the parity space method, the parameters and distur-

### REFERENCES


