A COMPARISON OF CYLINDRICAL PASTING METHODS

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Abstract: In this paper, we study six different boundary control point mappings for cylindrical surface pasting and compare the resulting pasted surfaces for C^0 join continuity. All six methods are algorithmically simple with low computational costs, requiring minimal computation aside from surface evaluation. The results demonstrate an order of magnitude quality improvement for some of our methods on a convex-only curved base, however, as the complexity of the base surface increases all methods show similar performance.

1 INTRODUCTION

The ability to construct smooth composite surfaces with multiple levels of control and adjustability, that can be modified or animated at interactive rates, is important to the modeling industry and to computer aided geometric design research. Hierarchical modeling offers a conceptual basis for generating surfaces with varying levels of detail. The surface building blocks are typically tensor product B-spline surfaces, preferred for their flexibility, compact representation and adjustable levels of internal continuity. Traditional methods for adding local detail to tensor product B-spline surfaces are knot insertion or degree raising, which increase complexity of the entire surface. Instead, hierarchical modeling techniques such as hierarchical B-splines (Forsey and Bartels, 1988), certain wavelet methods (Stollnitz et al., 1996), displacement mapping (Foley et al., 1990) and surface pasting (Bartels and Forsey, 1991; Barghiel et al., 1995) operate locally.

Surface pasting was introduced by Bartels and Forsey (Bartels and Forsey, 1991) to mimic the physical process of modeling with clay. It places a tensor product B-spline surface called the feature, on top of a tensor product B-spline surface hierarchy known as the base, via a smart mapping of feature control points onto the base. Surface pasting has a couple advantages over other known hierarchical modeling techniques — it offers lower computational costs, lower storage requirements, easy repositioning, and flexibility of non-parametric alignments. At the same time, because of the approximations involved in mapping a feature onto its base, a pasted feature is not guaranteed to meet its base surface with any order of continuity at the join boundary.

Original surface pasting focused upon constructing hierarchical surfaces comprised only of tensor product patches. The standard algorithm then utilized the linearity of patch boundary control points to minimize join discontinuities.

Cylindrical surface pasting was introduced by Mann and Yeung (Mann and Yeung, 2001) to extend the scope of pasting to model surfaces that include tensor product cylinders. In their work, the process of pasting a cylindrical feature's boundary onto a base surface was accomplished using the same approximations used by standard surface pasting. However, a fundamental construction difference between the closed curve boundary of a tensor product cylinder and the linear boundary of a tensor product patch leads us to believe that we can do better than this direct application.

In this paper, we propose and examine five alternative cylindrical pasting techniques that attempt to account for the cylindrical feature's structural difference. Previous work has been done to improve pasted surface quality by incorporating approximation schemes such as quasi-interpolation (Conrad and Mann, 2000), least-squares fittings (Leung and Mann, 2003) and Greville point interpolation (Siu and Mann, 2003); the first two of these extensions having only

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been examined in the context of patch pasting. In the context of cylindrical pasting these implementations are expected to suffer from high evaluation costs and/or significant algorithmic complexities. Our work focuses upon determining a cylindrical pasting technique that is computationally inexpensive, simple to implement, and that consistently gives the best comparative approximation of boundary continuity (i.e., C^0 continuity) between a pasted feature cylinder and its underlying base surface. In a more general sense, our work attempts to find a low-cost method of using a given closed B-spline curve to approximate a different given closed curve, with minimal reproduction error.

2 PASTING BASICS

We begin with an overview of surface pasting theory since it forms the foundation for the work done in this paper. Surface pasting combines a base surface and a feature surface, each of which is in tensor product B-spline form and defined over its own two-dimensional domain. The basic idea is to adjust the feature's control points in a manner that results in the boundary of the pasted feature lying on or near the base surface, while simultaneously ensuring that the shape of the pasted feature reflects characteristics of both its original form as well as of its base. To achieve this, the pasting process involves the following steps:

1. The feature's domain is embedded into its range space. Tensor product construction ensures that each feature control point $P_{i,j}$ has an associated domain point at which it maximally influences the feature surface. This domain point is referred to as the Greville point $(\gamma_i, \gamma_j) = \gamma_{i,j}$, where γ_i is the i^{th} Greville abscissa in the u parametric direction and γ_j is the j^{th} Greville abscissa in the v direction. Taking the embedded Greville point $(\gamma_{i,j}, 0) = \Gamma_{i,j}$ in feature range space as a point of origin, and using the feature's corresponding parametric domain directions to define a set of basis vectors, a local coordinate frame associated with each $P_{i,j}$ is constructed. Now, each feature control point can be expressed relative its local coordinate frame in terms of a displacement from its origin called the Greville displacement $\vec{d}_{i,j}$.
2. The feature domain is mapped into the base domain using an invertible transformation T. T determines the relative size and placement of the feature surface with respect to its base.
3. A base domain displacement representation of each feature control point is created by expressing each displacement $\vec{d}_{i,j}$ in terms of a local base coordinate frame.
4. The feature control points are positioned relative to the base surface using the local base coordinate frame and the mapped displacement vector recomputed relative to this frame.

Note that surface pasting is only an approximation technique. Rather than mapping every point of the feature surface, it maps a small number of sample sites, the feature Greville points. If the feature surface is described by too few control points or a coarse knot structure relative to its base, noticeable gaps at the join boundary may appear in the composite surface. In general, there is no guaranteed continuity between feature and base surfaces.

In the case of standard surface pasting, C^0 continuity is approximated by defining the embedded feature domain such that all boundary control points of the feature coincide with their respective Greville points. This ensures that the feature's boundary control points lie in the feature's domain plane and that upon being pasted they will lie directly on the base surface. Provided the base has low curvature relative to the spacing between these points, a near C^0 join is achieved. For further details on standard surface pasting, refer to earlier works on the subject (Bartels and Forsey, 1991; Barghiel et al., 1995).

Cylindrical surface pasting integrates concepts from parametric trimline-based blending to extend surface pasting to handle a wider variety of modeling situations. While standard pasting was designed only to handle the pasting of one open surface atop another, cylindrical pasting offers a method for connecting two base surfaces smoothly using a tensor product cylinder as the feature surface. In this paper, we are only concerned with pasting one end of a cylinder onto a base tensor product surface.

To paste one end of a cylindrical feature onto its base surface, the corresponding edge of the feature domain is mapped onto a paste curve in the base domain. Determining the placement of control points such that the pasted cylinder's boundary closely matches the image of a user-defined paste curve on the base surface is our challenge. The original cylindrical pasting technique directly applied the C^0 continuity approximation of standard surface pasting to the C^0 layer of a feature cylinder, i.e., each boundary control point was located at its corresponding Greville point. However, while zero displacement control points reproduce the linear boundary of a tensor product patch, placing the control points on the closed curve boundary of a tensor product cylinder does not reproduce its boundary (Figure 1). Our work explores alternate methods that have the potential of producing better C^0 continuity between the pasted cylinder and its underlying base. To maintain the prototyping nature of pasting, the methods we have designed have low computational costs, with a paste not costing much more than one base surface evaluation per boundary control point. We have limited our study to algorithmically simple techniques, as our intent was also to establish a standard for pasting the boundary of a cylindrical feature.

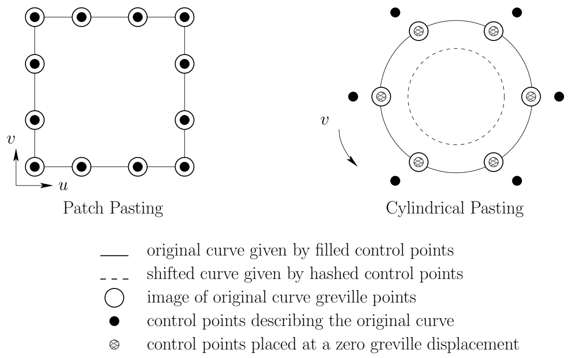

Figure 1: Greville displacement of boundary control points set to zero.

3 BOUNDARY CONTINUITY IN CYLINDRICAL PASTING

The results of pasting the boundary of a feature cylinder onto a base surface depend upon the feature-to-base space mappings used. We have examined four types of mappings (with minor variations on two of them), which we present here. Our method discussions assume that a given $m \times n$ tensor product feature cylinder $C(u,v) = \sum_{i=0}^{M}\sum_{j=0}^{N} P_{i,j}N_{i,j}(u,v)$, as shown in Figure 2, is being pasted onto the surface of a tensor product base $B(u,v)$ along the cylinder's $L_0 : u = u_0$ edge. The pasted feature boundary is to be constructed as an approximation to a curve on the base surface called the trim curve. The trim curve is a mapping into base range space of a user-defined circular paste curve given in the base domain. Although a polynomial paste curve could have been used, working with a circular representation allows for simpler implementations while still providing useful paste-quality comparisons (Aggarwal, 2004).

3.1 Greville Paste

The Greville Paste method is similar in concept to the technique described in the original work on cylindrical pasting. It assumes that the feature cylinder's boundary control points and its corresponding surface Greville points coincide, i.e., the boundary control points lie on the cylinder's edge. As discussed in §2, this assumption will inevitably result in a gap between the pasted cylinder boundary and the base surface trim curve. Although this cylindrical pasting C^0 continuity approximation seems less than acceptable, the Greville Paste method is useful as a comparative base case method. In particular, one can expect its application to yield three things:

1. a minimum cost metric: it requires only one surface evaluation per boundary control point
2. a maximum acceptable error bound
3. a well-defined convergence: upon infinite refinement the pasted control points will define the trim curve exactly.

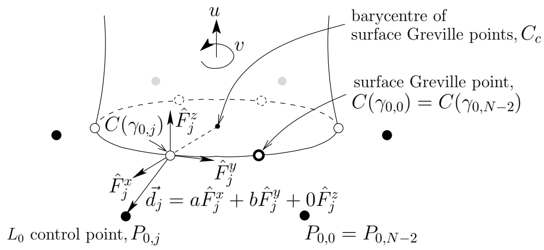

Figure 2: Feature Cylinder $C(u,v)$.

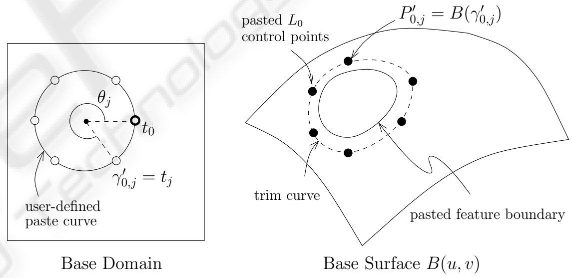

Figure 3: Greville Paste.

To describe the Greville Paste process we use Figure 3. We start by embedding the feature cylinder's L_0 surface Greville points $\{C(\gamma_{0,j})\}_{j=0}^{N-3}$ into the base domain. The embedded Greville points $\gamma'_{0,j}$ are obtained by a simple placement of each $\gamma_{0,j}$ onto a corresponding paste point t_j given on the paste curve in the base domain's uv-plane. The paste point t_0 is chosen relative to the centre of the paste curve circle at an angle of zero degrees to the u-parametric direction of the base domain, and the remaining t_js are then determined in proportion to the v-interval of the cylinder's domain. Now, each $C(\gamma_{0,j})$ can be mapped onto the base surface by performing a de Boor surface evaluation of the base at its embedded Greville point, giving the pasted Greville point $B(\gamma'_{0,j})$. Placing the feature cylinder's L_0 control points at the pasted Greville points with zero displacement results in a set of pasted control points $P'_{0,j} = B(\gamma'_{0,j})$, which are used to describe the cylinder's pasted boundary.

3.2 Control Point Paste

Control Point Paste is the first of three new cylindrical pasting techniques we devised, that attempt to better account for the non-zero Greville displacements of cylinder control points. Under Control Point Paste, the embedded location of each cylinder control point within the base domain is determined as the sum of its associated embedded Greville point and transformed Greville displacement vector. Implementation specifics are illustrated in Figure 4.

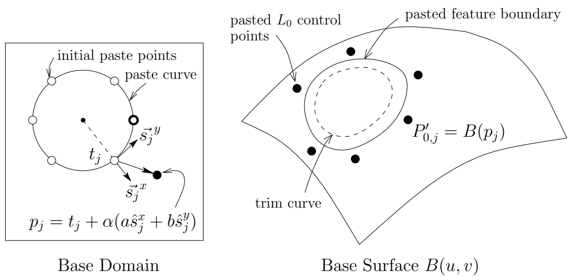

Figure 4: Control Point Paste.

The first step is to determine the L_0 Greville displacement vectors $\vec{d}_j$ (Figure 2). For each control point $P_{0,j}$, $\vec{d}_j$ is computed with respect to a unique local coordinate frame F_j, constructed such that

- $F_j^{\mathcal{O}}$, the origin of F_j, is at its corresponding surface Greville point: $F_j^{\mathcal{O}} = C(\gamma_{0,j})$
- $\hat{F}_j^x$ is given by a unit normal along the difference vector between the surface Greville point and the centre of the cylinder's L_0 edge (C_c, determined using Ceva's Theorem (Wells, 1991)) as: $\hat{F}_j^x = \frac{F_j^{\mathcal{O}} - C_c}{|F_j^{\mathcal{O}} - C_c|}$
- $\hat{F}_j^y$ is given by the normalized tangent to the cylinder's boundary curve at the chosen origin, and is along the v-parametric direction; this is also the directional derivative obtained by a de Boor evaluation of the L_0 curve
- $\hat{F}_j^z$ is given by a unit vector perpendicular to both $\hat{F}_j^x$ and $\hat{F}_j^y$.

The coordinates of each control point in relation to this local frame give the xyz components of the Greville displacements. By construction, the tensor product cylinders we use are such that the control points within each u-layer are coplanar, therefore, the z-component is always zero.

The initial paste points t_j on the base domain paste curve are determined as they were for the Greville Paste method. These are the locations at which we would like our mapped L_0 surface Greville points to lie. So, using our $\vec{d}_j$s we compute a relative placement of feature control points within the base domain space. In particular, $\hat{F}_j^x$ is mapped to $\hat{s}_j^x$, the out direction at t_j given by the 2D difference vector between t_j and the circular paste curve's centre point. $\hat{F}_j^y$ maps to $\hat{s}_j^y$ along the tangent to the paste curve at t_j. To account for the space change, a scale factor α, equal to the ratio of paste curve to cylinder curve radii is used. Applying the proportional displacement gives the paste points p_j within the base domain. De Boor evaluations at the p_js produce the set of pasted control points $P'_{0,j} = B(p_j)$ describing the pasted cylinder boundary using Control Point Paste.

Applying a 3D displacement within a 2D domain space results in pasted control points that lie on the base surface; however, the resulting pasted cylinder edge is unlikely to lie on the base unless the paste region is planar. A potential way to avoid errors introduced by 3D-in-2D computations is to account for the feature's L_0 Greville displacements in the 3D base range space instead. This alternative is explored using the next method.

3.3 Directional Displacement Paste

Directional Displacement Paste attempts to reduce C^0 gaps by computing the pasted control point locations relative to points on the trim curve in the base range space. The details, in context of Figure 5, follow.

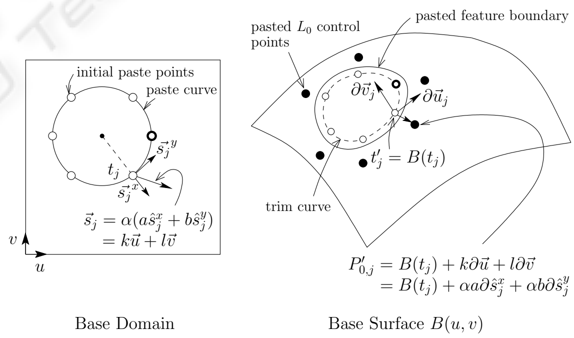

Figure 5: Directional Displacement Paste (Local).

Evaluating the base surface at points t_j generates a set of points t'_j lying on the trim curve. Ideally, the pasted cylinder boundary will be placed exactly on top of this trim curve. This suggests that the pasted cylinder edge should be constructed such that all the trim points t'_j lie on it. By definition, surface Greville points lie on the surface they describe. Therefore, Directional Displacement Paste maps the L_0 surface Greville points onto the trim points. The cylinder's pasted L_0 control points are then computed by placing them relative to these pasted Greville point locations. The displacements are determined using the L_0 Greville displacement vectors $\vec{d}_j$ (§3.2), which are mapped through the base domain ($\vec{s}_j$) onto the base surface. A point-vector addition of transformed feature-to-base space L_0 Greville points and Greville displacements gives the pasted cylinder's boundary control points.

A potentially useful modification to Directional Displacement Paste came about from observing the performance of the above described method on initial test data. Over a hump-like paste region, the high surface curvature at the trim points displaced the control points in a manner that pushed the pasted feature boundary well below the base surface trim curve. Therefore, we examined the pasting behavior when the displacements are computed using a simple approximation of average surface curvature over the paste region instead of local curvature.

To incorporate an average surface curvature, we perform a mapping of the paste curve's centre point onto the base, and compute base surface directional derivatives at it. The pasted boundary control point locations are then determined by applying the corresponding Greville displacement vector's components along the centre point's uv-directional vectors. The local directional derivatives at each t'_j no longer need to be computed. We refer to our original technique as Local Directional Displacement and the modified method as Average Directional Displacement.

3.4 Relative Displacement Paste

Relative Displacement Paste was motivated by Directional Displacement Paste. It too attempts to compute the L_0 pasted cylinder control points by accounting for Greville displacements in the base range space. The difference is in how the displacement frame is constructed at each mapped cylinder Greville point on the base. Directional Displacement mapped each feature surface displacement frame F_j onto the base surface via the 2D base domain space (Figure 5). However, this mapping does not maintain the relationship of displacement frame directions to corresponding feature boundary points when the shape of a cylinder's L_0 boundary curve distorts with pasting. Relative Displacement Paste maps the feature surface displacement frame directly onto the base surface so as to maintain the original F_j relationships to the feature boundary.

We describe this method in relation to Figure 6. For all $j = \{0, \ldots, N-3\}$, F_j, $\vec{d}_j$, $\gamma'_{0,j} = t_j$, and $t'_j = B(t_j)$ are computed exactly as for Directional Displacement Paste (§3.3). The pasting displacement frame S_j at each t'_j is then constructed such that

- $\hat{S}^x_j$ is the unit difference vector between t'_j and the new barycentre of pasted Greville points;

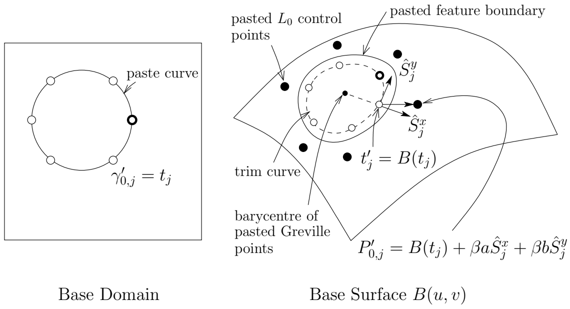

Figure 6: Relative Displacement Paste (Average).

- $\hat{S}^y_j$ is the normalized tangent to the trim curve at t'_j, given by the difference of slopes between t'_j and its two neighboring pasted Greville points.

The local frame directions $\hat{F}^x_j$ and $\hat{F}^y_j$ are now mapped to $\hat{S}^x_j$ and $\hat{S}^y_j$ respectively. The xy components of $\vec{d}_j$, i.e., a and b are applied along $\hat{S}^x_j$ and $\hat{S}^y_j$ respectively to give a control point placement relative t'_j. A change of space scale factor β is applied to the displacement vector to account for the transformation from feature surface space to base surface space. β is computed as the ratio of the average distance between original L_0 surface Greville points and their barycentre to the average distance between pasted surface Greville points and their barycentre.

Two variations of Relative Displacement Paste were examined — Average Relative Displacement uses the barycentre of all $\{t'_j\}_{j=0}^{N-3}$ to compute $\hat{S}^x_j$, whereas Local Relative Displacement uses a local barycentre given by t'_j and its two neighboring pasted Greville points. The rationale for the local method being that when the curvature of a base surface has more noticeable variations over the paste region boundary, locally affected $\hat{S}^x_j$s may offer a better placement of the control points.

4 COMPUTATIONAL ANALYSIS

The most significant computational cost in surface pasting is the number of base surface evaluations that need to be performed to position the feature's pasted control points. One surface position evaluation per L_0 cylinder control point is the minimum C^0 requirement for any cylindrical pasting method. An approximation improvement technique such as knot insertion rapidly becomes unacceptable as it doubles the number of control points at each level of refinement, increasing evaluation costs exponentially. The methods we have described in this paper attempt to improve a feature cylinder's pasted boundary approximation of the trim curve for relatively small increases in cost. A comparative summary of pasting costs per boundary control point is given in Table 1 (refer to (Aggarwal, 2004) for a detailed analysis). It may be noted that the cost of pasting a control point using any of our methods is at most half as expensive as doubling the number of control points using knot insertion to improve the pasted join accuracy. To keep the relative costs in perspective, we note that a de Boor position-only surface evaluation for a bicubic tensor product surface requires 30 affine combinations, a position-with-derivatives evaluation takes 37 affine combinations, and a vector difference of a pair of points is one affine combination.

5 ERROR BOUNDS

To provide a bound on how fast the error in C^0 continuity is expected to converge with feature cylinder refinement, we use the concept of linear reproduction. Given a polynomial function F and its approximation P, a Taylor series expansion gives the error as $\sum_{i=0}^{\infty} \frac{F^{(i)}(\xi)-P^{(i)}(\xi)}{i!} h^i$. An interpolation method is said to have linear precision if $F(\xi) = P(\xi)$ and $F'(\xi) = P'(\xi)$. In this case, the first two terms of the Taylor series cancel, leaving an error of $\sum_{i=2}^{\infty} \frac{F^{(i)}(\xi)-P^{(i)}(\xi)}{i!} h^i = O(h^2)$, where h is the distance between samples. Standard surface pasting is expected to have this property, as was verified empirically by Conrad (Conrad, 1999).

Further mathematical analysis, as given in (Aggarwal, 2004), enables us to determine whether linear reproduction also holds for the cylindrical surface pasting methods described in this paper. Essentially, for each scheme, we examine whether the pasted cylinder's boundary is expected to be identical to its trim curve on a linear base surface. The results of our theoretical analysis are summarized in Table 1.

6 RESULTS

The empirical error between the feature cylinder's pasted boundary and the base surface trim curve offers an important comparison metric for evaluation of the feature-on-base boundary quality. For our error analysis, we examined the maximum position difference between these two closed curves, and the progressive refinement ratio of their differences describing the rate of error convergence. Our cylindrical boundary pasting schemes were evaluated for three different base surfaces of increasing complexity — a planar base, a simple curved base, and a base with an inflection. All the surfaces used (bases and feature cylinder) were bicubic surfaces. We chose circular paste curves defined by a center and radius in the base domain. Also, our feature was constructed to have a close-to-circular L_0 boundary.

C^0 continuity sampling information was generated by sampling the pasted feature boundary at 10 different positions for each non-overlapping domain interval in the cylinder's v-parametric direction. These points were compared against samples on the base trim curve taken at points associated with each v-parameter value of the cylinder domain. With each level of feature refinement, the number of samples taken was doubled.

Pasting the cylinder's boundary onto a planar base patch yielded a close to zero error for all methods except Greville Paste. This is in keeping with the theoretical expectation of linear reproduction for all methods other than Greville Paste.

Pasting onto a curved, convex-only, bicubic base that did not have any regions of negative Gaussian curvature, gave results supporting quadratic error convergence for all methods including Greville Paste. In this case, Average Directional Displacement and both Relative Displacement Paste methods were found to perform a magnitude better than Greville Paste.

For our final test case, we pasted the boundary of our feature cylinder onto a curved bicubic base over a region of negative Gaussian curvature. Results are given in Table 2 (Figure 7). The Relative Displacement Paste methods appear to perform an order of magnitude better when the feature knot structure is approximately as coarse as that of the base. However, one level of cylinder refinement improves Greville Paste to be comparable to the Relative Displacement techniques. The minor error reductions offered by the alternatives to Greville Paste are clearly offset by their extra computational costs. Further, even-though Greville Paste doesn't have the linear reproduction property, all our results indicate that it has $O(h^2)$ error convergence. Therefore, it appears that as the complexity of the base surface increases, the C^0 pasting results obtained using Greville Paste are comparable to all our other cylindrical pasting approaches.

7 CONCLUSION

In this paper, we have examined six different control point placements for describing the pasted cylinder boundary – Greville Paste, Control Point Paste, Local Domain Displacement, Average Domain Displacement, Local Relative Displacement, and Average Relative Displacement. Greville Paste was essentially a direct application of the standard surface pasting algorithm to cylindrical pasting.

Greville Paste seemed intuitively inadequate for

Table 1: Costs and linear reproducibility associated with pasting a $m \times 3$ cylinder onto a bicubic patch. N is the number of control points.

Method	Affine combinations per control point	Linear reproduction satisfied
Greville Paste	30	no
Control Point Paste	40	yes
Local Directional Displacement	47	yes
Average Directional Displacement	$40+\frac{37}{N-3}, N \geq 9$	yes
Local Relative Displacement	44	not necessarily
Average Relative Displacement	43	yes

pasting cylinders because its placement of pasted control points onto the desired join boundary could never reproduce the corresponding closed curve. Based upon the theory used in constructing our methods, we expected that Local Domain Displacement would result in the most accurate C^0 paste for any base surface irrespective of its complexity. Instead, an empirical analysis of the error between the pasted cylinder boundaries and the desired trim curve indicates that the less-intuitive Relative Displacement Paste methods most consistently produce the best quality join. However, the relative improvement over Greville Paste drops rapidly with every level of cylinder refinement when pasting onto a complex base surface.

It comes as a surprise that, in general, Greville Paste does as well as any of our other cylindrical boundary pasting methods. Our results further confirm that the best possible error convergence offered by the methods explored is quadratic in all cases. Therefore, the original standard pasting technique is a reasonable standard not only for patches, but also for cylinders.

8 FUTURE WORK

Our work focused on establishing a simple and low-cost cylindrical boundary pasting standard. It is recommended that another study be performed to assess the results of cylindrical boundary pasting using methods such as quasi-interpolation, least-squares fitting, and Greville point interpolation. Although the initial computation costs are expected to be notably high for these methods, they may be effectively compensated for by low re-evaluation costs when pasting over the same region. The algorithmic complexity of these alternate methods may still be a concern.

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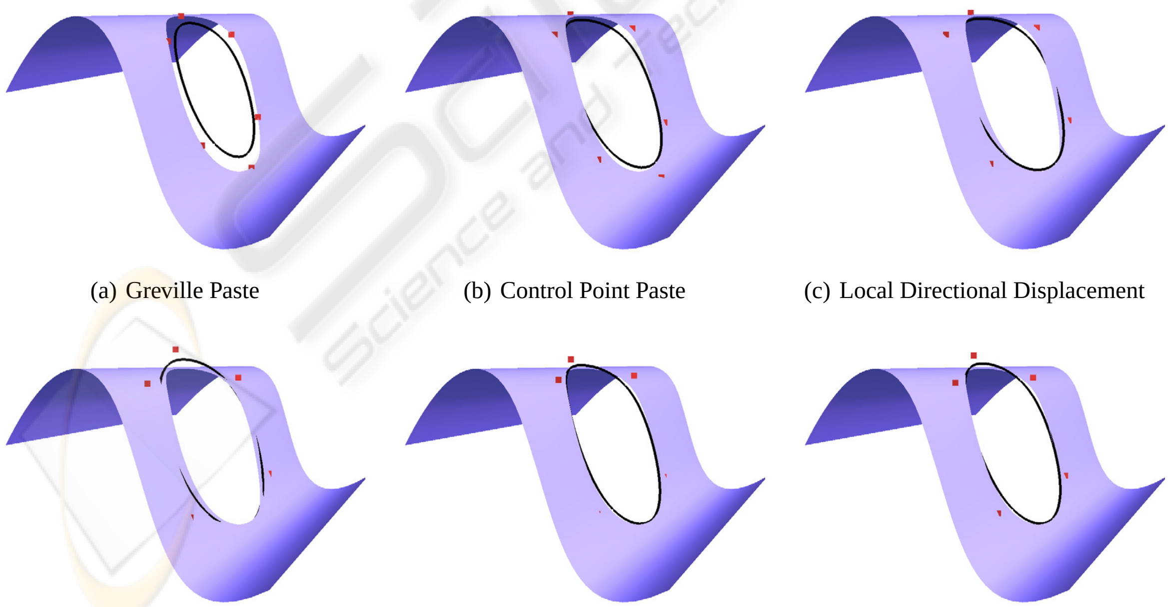
Table 2: Experimental errors from pasting a feature cylinder's boundary onto a bicubic base over a region of negative Gaussian curvature. The ratio in row i is the ratio of Max_i to Max_{i-1} for the corresponding method.

	Greville Paste		Control Point Paste		Local Directional Displacement	
Refinement	Max	Ratio	Max	Ratio	Max	Ratio
0	0.185416	na	0.067727	na	0.131738	na
1	0.049609	3.74	0.023591	2.87	0.037600	3.50
2	0.012585	3.94	0.006567	3.59	0.009219	4.08
3	0.003157	3.99	0.001600	4.10	0.002005	4.60
4	0.000790	4.00	0.000359	4.46	0.000473	4.23
5	0.000198	4.00	0.000089	4.05	0.000114	4.15

	Avg Directional Displacement		Local Relative Displacement		Avg Relative Displacement	
Refinement	Max	Ratio	Max	Ratio	Max	Ratio
0	0.123581	na	0.037246	na	0.039474	na
1	0.035348	3.50	0.014900	2.50	0.018923	2.09
2	0.008783	4.02	0.005125	2.91	0.004766	3.97
3	0.001984	4.43	0.001332	3.85	0.001232	3.87
4	0.000471	4.21	0.000299	4.45	0.000276	4.47
5	0.000113	4.18	0.000073	4.10	0.000067	4.10



(a) Greville Paste (b) Control Point Paste (c) Local Directional Displacement

(d) Average Directional Displacement (e) Local Relative Displacement (f) Average Relative Displacement

Figure 7: Pasting onto a curved bicubic base over a region of negative Gaussian curvature.