ON THE STABILITY OF THE DISCRETE TIME JUMP LINEAR SYSTEM

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Abstract: In this paper we investigate the relationships between individual mode stability and mean square stability of jump linear system. It is well known that generally stability of a dynamical system in all its modes does not guarantee stability of the jump linear system defined by all these modes. We present conditions under which stability of all modes implies the mean square stability of the overall system.

1 INTRODUCTION

Linear dynamical systems with Markovian jumps in parameter values have recently attracted a great deal of interest. The main reason is that such systems may serve as models for a large variety of industrial control processes. One class of examples is given by nonlinear control plants characterized by linearized models corresponding to several operating points which appear due to abrupt changes. Such problems are typical in control systems of a solar thermal central receiver (Sworder and Rogers, 1983), electric load modeling (Malhame and Chong, 1985), aircraft flight control systems (Moerder et al., 1989) or ship autopilot systems (Astrom and Wittenmark,1989). Markovian jumps in parameter values may also result from the random failure/repair processes(see e.g. (Siljak, 1980), (Willsky, 1976), (Petkovski, 1987)) and resulting fault prone control systems ((Siljak, 1980), (Siljak, 1980), (Swierniak et al., 1998)). Yet another class of processes with Markov jumps could be met in flexible manufacturing systems (see e.g. (Boukas and Haurie, 1990), (Boukas et al., 1995), (Boukas and Liu, 2001)). Moreover in (Athans, 1987) Athans suggested that such a model setting has also the potential to become a basic framework in solving control-related issues in battle management, command, and communications systems.

This paper deals with the problem of stability of the linear discrete time dynamical systems with Markovian jumps in the parameters values. This problem has been investigated in a number of papers (see e. g. (Fang and Loparo, 2002), (Ji and Chizeck, 1990a), (Feng et al., 1992), (Li et al., 2000)) not only because of its theoretical complexity but also because its broad practical implications. The way of modeling is to represent the overall system by a number of deterministic linear plants and a Markov chain the states of which called modes define which among these deterministic plants describes the dynamics of the system in given time. The interesting but well known property of the jump linear systems (see e.g. (Fang and Loparo, 2002)) is that deterministic stability of all system matrices in all modes is neither necessary nor sufficient condition for mean square stability of the considered process.

More precisely let us fix an underlying probability space $\{\Omega, \mathcal{F}, P\}$ and consider the linear system with Markovian jumping parameters

$$x\,(n+1) = A(r(n))x\,(n)\,, \qquad (1)$$

or it controlled analogue

$$x\,(n+1) = A(r(n))x\,(n) + B(r(n))u\,(n)\,, \qquad (2)$$

where $x(n) \in R^l$ denotes the state vector, $u(n) \in R^m$ is the control input, $r(n)$ is a Markov chain which takes values in a finite set $S = \{1, 2, ..., s\}$ with transition probability matrix $P = [p\,(i,j)]_{i,j\in S}$ and initial distribution $P\,(r\,(0) = i_0) = 1$. Furthermore, for $r(n) = i$, $A_i := A(i)$ and $B_i := B(i)$ are constant matrices of appropriate sizes. Denote by $x\,(n, x_0, i_0)$ the solution of (1) with initial condition $x(0) = x_0$ and initial distribution $P\,(r\,(0) = i_0) = 1$ at time $n = 0$. The control $u\,(n)$ is assumed to be

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measurable with respect to the $\sigma-$field generated by $\{r(0), ..., r(n)\}$.

In the literature three types of stability of system (1) are considered: mean square stability, moment stability and almost sure stability. In this paper we investigate the mean square stability and stabilizability. Their formal definitions are as follows.

Definition 1 *System (1) is mean square stable(MS stable), if for all $(x_0, i_0) \in R^l \times S$, we have*

$$\lim_{n \longrightarrow \infty} E \left\| x\left(n, x_0, i_0\right) \right\|^2 = 0.$$

In that case we call the pair (A, P), where $A = (A_1, ..., A_s)$ mean square stable. System (2) is mean square stabilizable (MS stabilizable) if there exists a feedback control $u(n) = K(r(n))x(n)$ such that the resulting closed loop system is mean square stable. In that case we call (A, B, P) MS stabilizable, where $B = (B_1, ..., B_s)$.

The natural question which we have mentioned at beginning is how the individual properties of matrices A_i or pairs of matrices (A_i, B_i) such as deterministic stability or deterministic stabilizability (controllability) are related to the MS stability of (1) or MS stabilizability of (2).

It is known that deterministic stability of all matrices A_i, $i \in S$ is neither necessary nor sufficient for MS of (1) (see (Ji and Chizeck, 1990b), Examples 6.1). However the deterministic stability of each matrices $\sqrt{p(i,i)}A_i$ is necessary for MS stability (see Theorem 2.1 in (Ji and Chizeck, 1990b)). It is also known that individual deterministic stability of each pair (A_i, B_i) is neither sufficient nor necessary (see Example 6.2 in (Ji and Chizeck, 1990b)) for MS stabilizability of (2). Moreover the individual deterministic controllability of each pair (A_i, B_i) is not sufficient for MS stabilizability of (2) (see Example 6.2 in (Ji and Chizeck, 1990b)).

It is also known that if the switching signal r in (1) is deterministic and all matrices A_i, $i \in S$ are stable, then it is possible to ensure the stability of the system by switching sufficiently slowly. This means that instability arises in (1) as a result of rapid switching. Given this basic fact, a natural and obvious method to ensure the stability of (1) is to somehow constrain the rate at which switching takes place. This idea has appeared in many studies on time varying systems (see, (Guo and Rugh, 1995), (Ilchmann et al., 1987)).

In this paper we consider the system (1) with matrices A_i, $i \in S$ being stable and investigate the problem how to characterize the transition probabilities P for which the system is MS stable. We also propose certain new sufficient conditions for MS stability of (1) in terms of spectra of matrices A_i and basing on this result we propose new sufficient conditions for MS stabilizability of (2) given in terms of individual pair of matrices (A_i, B_i). Finally we present a procedure for MS stabilization of system (2) by pole placement of pairs (A_i, B_i).

2 STABILITY OF MATRICES A_I VERSUS MS STABILITY

The necessary and sufficient conditions for mean square stability of (1) are given by the following theorem (see, (Costa and Fragoso, 1993)).

Theorem 1 *The system (1) is mean square stable if and only if there exists a positive definite solution P_i, $i \in S$ of the following coupled Lyapunov inequality*

$$P_i - A_i' \left(\sum_{j \in S} p_{ij} P_j \right) A_i > 0.$$

For a symmetric matrix X of size n by n denote $\lambda_1(X), ..., \lambda_n(X)$ the eigenvalues arranged such that $\lambda_1(X) \geq ... \geq \lambda_n(X)$. We have the following theorem.

Theorem 2 *If matrices A_i, $i \in S$ are such that $\lambda_1 \left(A_i A_i'\right) < 1$, and there exists a sequence of positive numbers $(x_i)_{i \in S}$ such that*

$$\frac{x_i}{\lambda_1 \left(A_i A_i'\right)} > \sum_{\substack{j \in S \\ j \neq i}} \frac{p_{ij} x_j}{\left(1 - \lambda_1 \left(A_j A_j'\right)\right)}, \; i \in S. \quad (3)$$

Then the system (1) is mean square stable.

Proof. Define the following n by n matrices

$$Q_i = x_i \cdot I.$$

If $\lambda_1 \left(A_i A_i'\right) < 1$, $i \in S$ then by inequality

$$\left|\lambda_1 \left(A_i\right)\right| \leq \sqrt{\lambda_1 \left(A_i A_i'\right)},$$

(see (Weyl, 1949)) $\left|\lambda_1 \left(A_i\right)\right| < 1$, matrices A_i are stable and consequently discrete Lyapunov equation

$$P_i - A_i' P_i A_i = Q_i, \quad (4)$$

has a positive definite solution P_i. Moreover the assumption that $\lambda_1 \left(A_i A_i'\right) < 1$ implies that the solution has the following upper bound (see (Yasuda and Hirai, 1979))

$$\lambda_1 \left(P_i\right) \leq \frac{\lambda_1 \left(Q_i\right)}{1 - \lambda_1 \left(A_i A_i'\right)}.$$

Using this inequality with the fact that $\lambda_1 \left(Q_j\right) = x_j$ we have

$$\lambda_1 \left(P_j\right) \leq \frac{x_j}{1 - \lambda_1 \left(A_j A_j'\right)}.$$

Therefore

$$\sum_{\substack{j\in S\\ j\neq i}} p_{ij}P_j \geq \sum_{\substack{j\in S\\ j\neq i}} \frac{p_{ij}x_j}{1-\lambda_1\left(A_jA_j'\right)}\cdot I$$

and

$$A_i'\left(\sum_{\substack{j\in S\\ j\neq i}} p_{ij}P_j\right)A_i \geq$$

$$\left(\sum_{\substack{j\in S\\ j\neq i}} \frac{p_{ij}x_j}{1-\lambda_1\left(A_jA_j'\right)}\right)\lambda_1\left(A_iA_i'\right).$$

Using the last inequality together with (4) we have

$$P_i - A_i'\left(\sum_{j\in S} p_{ij}P_j\right)A_i = P_i - p_{ii}A_i'P_iA_i -$$

$$A_i'\left(\sum_{\substack{j\in S\\ j\neq i}} p_{ij}P_j\right)A_i \geq$$

$$P_i - A_i'P_iA_i - A_i'\left(\sum_{\substack{j\in S\\ j\neq i}} p_{ij}P_j\right)A_i \geq\geq$$

$$x_i\cdot I - A_i'\left(\sum_{\substack{j\in S\\ j\neq i}} p_{ij}P_j\right)A_i$$

$$\left(x_i - \lambda_1\left(A_iA_i'\right)\sum_{\substack{j\in S\\ j\neq i}} \frac{p_{ij}x_j}{1-\lambda_1\left(A_jA_j'\right)}\right)\cdot I$$

and from the assumption 3 we conclude that

$$P_i - A_i'\left(\sum_{j\in S} p_{ij}P_j\right)A_i > 0,$$

and on the basis of Theorem 1, (1) is MS stable. ■

Using the Theorem 2 with $x_i = 1$, $i \in S$ and observe that

$$\sum_{\substack{j\in S\\ j\neq i}} \frac{p_{ij}}{\left(1-\lambda_1\left(A_jA_j'\right)\right)} \leq$$

$$\left(\sum_{\substack{j\in S\\ j\neq i}} p_{ij}\right)\max_{\substack{j\in S\\ j\neq i}}\left(\frac{1}{1-\lambda_1\left(A_jA_j'\right)}\right) =$$

$$\left(1-p_{ii}\right)\frac{1}{\min\limits_{\substack{j\in S\\ j\neq i}}\left(1-\lambda_1\left(A_jA_j'\right)\right)}$$

$$= \frac{1-p_{ii}}{1-\max\limits_{\substack{j\in S\\ j\neq i}}\lambda_1\left(A_jA_j'\right)}$$

we obtain the following Corollary.

Corollary 3 *If matrices A_i, $i \in S$ are such that $\lambda_1\left(A_iA_i'\right) < 1$, and*

$$\frac{\lambda_1\left(A_iA_i'\right) + \max\limits_{\substack{j\in S\\ j\neq i}}\lambda_1\left(A_jA_j'\right) - 1}{\lambda_1\left(A_iA_i'\right)} < p_{ii},\ i \in S. \tag{5}$$

Then the system (1) is mean square stable.

In the next theorem we will propose a design procedure for MS stabilization of system (2) under the additional constrains on the transition rates matrix. However it is worth to notice another possible application of this result. Sometimes (see (Ji and Chizeck, 1989) for details) it is reasonable to consider the system (2) with matrix P which is separately controlled. In that case the Theorem 2 may be used to find the values of P for which the system is MS stable.

As a straightforward consequence of Theorem 2 and the definition of MS stabilizability we have the following result.

Theorem 4 *Suppose that for system (2) there exist positive numbers $(x_i)_{i\in S}$ and feedback matrices $(K_i)_{i\in S}$, such that*

$$\frac{x_i}{\lambda_1\left(\widetilde{A}_i\widetilde{A}_i'\right)} > \sum_{\substack{j\in S\\ j\neq i}} \frac{p_{ij}x_j}{\left(1-\lambda_1\left(\widetilde{A}_j\widetilde{A}_j'\right)\right)},\ i \in S,$$

where $\widetilde{A}_i = A_i + K_iB_i$, then (A, B, P) is MS stabilizable and $(K_i)_{i\in S}$ is a stabilizable feedback.

3 CONCLUSIONS

We have discussed links between individual mode stability (stabilizability) and MS stability (MS stabilizability) of jump linear system. The main result states that if the individual mode are stable then the jump linear system is mean square stable provided that the probabilities p_{ii} are close to 1. It means that it is possible to ensure the stability of the jump system by switching sufficiently slowly between individual modes. Another consequence of the main result is a procedure to stabilize the jump linear system by pole placement of system matrices for all modes.

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