Keywords: Analysis, boundedness, composition, liveness, Petri nets, regularity.

Abstract: By the linear algebraic representation of Petri nets, Desel introduced regularity property (Desel, 1992). Regularity implies a sufficient condition for a Petri net to be live and bounded. All the conditions checking the regularity of a Petri net are decidable in polynomial time in the size of a net (Desel and Esparza, 1995). This paper proves that regularity, liveness and boundedness can be preserved after applying many compositional operations to Petri nets. This means that, by applying these compositional operations, a designer can construct complex nets satisfying regularity, liveness and boundedness properties from simpler ones without forward analysis.

1 INTRODUCTION

As a graphical and mathematical tool, Petri nets provide a uniform environment for modeling, formal analysis, and design of discrete event systems. One of the major advantages of using Petri net models is that the same model can be used for the analysis of behavioural properties and performance evaluation, as well as for systematic construction of discrete-event simulators and controllers (Zhou and Venkatesh, 1999).

One net is regular if it satisfies the conditions of the Rank Theorem described in algebraic methods (Desel, 1992; Desel and Esparza, 1995). Regularity is a sufficient condition for ordinary nets to be live and bounded (Desel and Esparza, 1995). In general, a system is live if all its operations are eventually executable, starting not only from its initial state but also from any reachable state. A system is bounded if it has a finite number of states. In the terminology of Petri nets, liveness requires the firability of every transition starting from any reachable marking, boundedness implies that the number of tokens existing in every place will not exceed a certain limit.

For system designs specified in Petri nets, the major approaches for verification include reachability analysis, direct proving on the basis of definitions, mathematical programming, characterization and property-preserving transformation. This paper relates to the fourth approach, i.e., the property-preserving transformation. In this approach, the original net is assumed to satisfy some specific properties and the transformation is required to preserve these properties in the transformed net. The advantage of this approach is that the transformed net is automatically correct without the need of forward verification.

Transformations on Petri nets may be roughly classified into three groups, namely reduction, refinement and composition. In the literature, there has been much work related to transformations that preserve liveness and/or boundedness (Berthelot, 1986; Berthelot, 1987; Esparza, 1994; Koh and DiCesare, 1991; Suzuki and Murata, 1983; Valette, 1979; Souissi, 1991; Zhou, 1996; Huang, Jiao and Cheung, 2005). The preservation of regularity has not been considered. This paper introduces four kinds of composition operations in terms of places and transitions. For each kind of the four composition operations, this paper proves that regularity, liveness and boundedness can be preserved automatically or under some simpler conditions.

This paper is organized as follows: Section 2 presents some basics about Petri nets including algebraic characterizations. Four compositional operations in terms of places and transitions are
introduced and the preservation of regularity is verified in Sections 3. Section 4 proves that liveness and boundedness can be preserved under these operations. In this section, an example is given to illustrate some results of this paper. Some concluding remarks are given in Section 5.

2 PRELIMINARIES OF PETRI NETS

This section outlines the definitions, terminology and properties as required in the paper.

A net is denoted by \( N = (P, T, F) \), where \( P \) is a non-empty finite set of places, \( T \) is a non-empty finite set of transitions with \( P \cap T = \emptyset \) and \( F \subseteq (P \times T) \cup (T \times P) \) is a flow relation. The pre-set of \( x \) is defined as \( x^- = \{ y \in P \cup T \mid (y, x) \in F \} \) and the post-set of \( x \) is defined as \( x^+ = \{ y \in P \cup T \mid (x, y) \in F \} \). Similarly, for any subset of \( Y \subseteq P \cup T \), \( Y^- \) (resp., \( Y^+ \)) denotes the union of all \( y^- \) (resp., \( y^+ \)) for all \( y \in Y \). A net \( N = (P, T, F) \) is said to be pure or self-loop-free iff \( x^- \cap x^+ = \emptyset \forall x \in P \cup T \). We just discuss pure nets in this paper.

The incidence matrix \( V \) of a pure net \( N \) is a \( |P| \times |T| \) matrix whose element \( v_{ij} \) at row \( p_i \) and column \( t_j \) is denoted as follows: \( v_{ij} = 1 \) if \( p_i \in t_j^* \); \( v_{ij} = -1 \) if \( p_i \in t_j^* \); and \( v_{ij} = 0 \) if \( p_i \notin t_j^* \).

A marking of a net \( N = (P, T, F) \) is a mapping \( M: P \rightarrow \{0, 1, 2, \ldots \} \). A place \( p \) is said to be marked by \( M \) if \( M(p) > 0 \). A transition \( t \) is enabled at a marking \( M \) if for every \( p \in t^* \), \( M(p) \geq 1 \). A transition \( t \) may be fired if it is enabled. Firing a transition \( t \) results in changing the marking \( M \) to a new marking \( M' \), where \( M' \) is obtained by removing one token from each \( p \in t^* \) and by putting one token to every \( p \in t^* \). \( R(N, M_0) \) denotes the set of all markings reachable from the initial marking \( M_0 \).

A transition \( t \) is said to be live in a Petri net \( (N, M_0) \) iff, for any \( M \in R(N, M_0) \), there exists \( M' \in R(N, M) \) such that \( t \) can be fired at \( M' \). \( (N, M_0) \) is said to be live iff every transition of \( N \) is live. A place \( p \) is said to be bounded in \( (N, M_0) \) iff there exists a constant \( k \) such that \( M(p) \leq k \) for all \( M \in R(N, M_0) \). \( (N, M_0) \) is bounded iff every place of \( N \) is bounded.

For \( x \in P \cup T \), the cluster of \( x \), denoted as \( [x] \), is the smallest subset of \( P \cup T \) satisfying three conditions: (1) \( x \in [x] \); (2) if \( p \in P \cap [x] \) then \( p^* \subseteq [x] \); and (3) if \( t \in T \cap [x] \) then \( t^* \subseteq [x] \). \( N \) is said to satisfy the rank-and-cluster property iff the rank of its incidence matrix is less than the number of its clusters by 1.

A net \( N \) is said to be connected iff every pair of nodes \((x, y)\) satisfies \((x, y) \in (F \cup F^{-1})^*\). A net \( N \) is said to be strongly connected iff \((x, y) \in F^*\), i.e., there exists a directed path from every node \( x \) to every node \( y \). A P-invariant (resp., T-invariant) of \( N \) is a non-negative integer \(|P|\)-vector \( \alpha \) (resp., \(|T|\)-vector \( \beta \)) satisfying the equation \( \alpha V = 0 \) (resp., \( V \beta^T = 0 \)), where \( V \) is the incidence matrix of \( N \). A P-invariant \( \alpha \) (resp., T-invariant \( \beta \)) of a net is called semi-positive if \( \alpha \geq 0 \) and \( \alpha \neq 0 \) (resp., \( \beta \geq 0 \) and \( \beta \neq 0 \)). The support of a semi-positive P-invariant \( \alpha \), denoted by \( \langle \alpha \rangle \), is the set of places \( p \) satisfying \( \alpha(p) > 0 \), and the support of a semi-positive T-invariant \( \beta \), denoted by \( \langle \beta \rangle \), is the set of transition satisfying \( \beta(t) > 0 \) (Desel and Esparza, 1995).

A net \( N \) is regular (Desel 1992) iff (1) \( N \) is connected, (2) \( N \) has a positive P-invariant, (3) \( N \) has a positive T-invariant, and (4) \( N \) satisfies the rank-and-cluster property.

In \( N \), a non-empty set of places \( D \) is said to be a siphon (resp., trap) iff \( D \subseteq D^* \) (resp., \( D'^* \subseteq D \)). A siphon (resp., trap) is said to be minimal if it does not properly contain any other siphon (resp., trap).

For more details, please refer to (Recalde, Teruel and Silva, 1998; Silva, Teruel and Colom, 1998).

3 FOUR COMPOSITIONAL OPERATIONS AND THE PRESERVATION OF REGULARITY

This section considers four compositional operations in terms of places and transitions. Two of them are very natural and can be found in the literature. The other two operations are a little similar to those in (Berthelot, 1987). Suppose the original nets are regular. We will prove that, for two of the four compositional operations, the regularity can be automatically preserved. For the other two ones, some simple conditions will be provided under which the regularity can be preserved.

3.1 Merging a Pair Of Places From Two Nets

COMPOSITION-BY-PLACE (composition via merging a pair of places from two different nets): Consider two disconnected ordinary nets \( N_1 = (P_1 \cup \)
such that and . Let be composed from and by merging the pair of places and into , that is, . Let be composed from and by merging the pair of places and into , that is, . Let be composed from and by merging the pair of places and into , that is, . Let be composed from and by merging the pair of places and into , that is, .

**Theorem 1:** Let , and be defined in COMPOSITION-BY-PLACE. Then, is regular if and are regular.

**Proof:** In order to show that is regular, we must prove that is connected, has a positive -invariant and a positive -invariant, and satisfies: \( \text{Rank}(N) = |C| - 1 \). The incidence matrices of , and have the forms:

\[
V = \begin{pmatrix}
T_1 & T_2 \\
V_{11} & 0 \\
V_{21} & V_{22}
\end{pmatrix}
\]

\[
V_1 = \begin{pmatrix}
T_1 & T_2 \\
V_{11} & V_{21} & V_{22}
\end{pmatrix}
\]

\[
V_2 = \begin{pmatrix}
T_1 & T_2 \\
V_{11} & V_{21} & V_{22}
\end{pmatrix}
\]

(1) Since and are connected, it is obvious that is also connected.

(2) Since and have positive -invariants, there exist and such that . Then, \( \alpha/Z = 0 \) and \( \alpha/V = 0 \). Let \( \alpha = (\alpha(p_2)\alpha(p_1)\alpha(p_3)\alpha(p_4)\alpha(p_5)\alpha(p_6)\alpha(p_7)) \). Then, \( \alpha > 0 \) and \( \alpha/V = (\alpha(p_2)\alpha(p_1)V_{11} + \alpha(p_1)\alpha(p_2)V_{31} + \alpha(p_1)\alpha(p_2)V_{32} + \alpha(p_1)\alpha(p_2)V_{33}) = (\alpha(p_2)\alpha(p_1)V_{11} + \alpha(p_1)\alpha(p_2)V_{31} + \alpha(p_1)\alpha(p_2)V_{32} + \alpha(p_1)\alpha(p_2)V_{33}) = 0 \). This means that is a positive -invariant of .

(3) Since and have positive -invariants, there exist and such that and . Then, \( V_1 = 0, V_2 = 0 \) and \( V_3 = 0 \). Let \( \beta = (\beta_1, \beta_2) \), then \( V_1 = (V_1\beta_1^T + V_2\beta_2^T + V_3\beta_3^T + V_4\beta_4^T) = 0 \). That is, is a T-invariant of .

(4) Since \( \text{Rank}(V_1) = |C(N)| - 1 \) and \( \text{Rank}(V_2) = |C(N)| - 1 \), then \( \text{Rank}(V_3) \leq |P_1| \) and \( \text{Rank}(V_4) \leq |P_2| \). Hence, the bottom row is a linear combination of the other rows of \( V_2 \) and \( \text{Rank}(V) = \text{Rank}(V_2) \) for \( i = 1 \) and . This also implies that the bottom row of \( V \) is a linear combination of the other rows of \( V_1 \) and \( \text{Rank}(V) = \text{Rank}(V_1) + \text{Rank}(V_2) = (|C(N)| - 1) + (|C(N)| - 1) \).

**3.2 Merging Two Non-neighboring Places In A Net**

**MERGE-N-PLACE** (merging two non-neighboring places in a net): Let a net \( N = (P_0 \cup \{p_1, p_2\}, T, F) \) be a net and \( p_1 \) and \( p_2 \) satisfy: \( \langle p_1 \cup p_2 \rangle \in \{ p_1 \cup p_2 \} = \emptyset \). Let \( N' = (P_0 \cup \{p_1, p_2\}, T, F') \) be obtained from \( N \) by merging the places \( p_1 \) and \( p_2 \) into a single place , where \( F' = F \cup \{ (t, p_1) | (t, p_1) \in F \text{ or } (t, p_2) \in F \} \cup \{ (p_1, t) | (p_1, t) \in F \text{ or } (p_2, t) \in F \} = \{ (t, p_i) | (t, p_i) \in F \text{ for } i = 1, 2 \} \). Let the incidence matrices of \( N \) and \( N' \) have the following forms:

\[
P_0 = \begin{pmatrix}
T_0 & T \\
V_0 & 0
\end{pmatrix}
\]

\[
P_1 = \begin{pmatrix}
T_1 & T_2 \\
V_{11} & V_{12} & V_{21} & V_{22}
\end{pmatrix}
\]

\[
P_2 = \begin{pmatrix}
T_1 & T_2 \\
V_{11} & V_{12} & V_{21} & V_{22}
\end{pmatrix}
\]

(1) Since is regular, is connected. It is obvious that is also connected.

(2) Since there exists a positive -invariant of \( N \) such that \( \alpha(p_1) = \alpha(p_2) \).

(3) \( p_1 \) and \( p_2 \) belong to the same cluster, and

(4) there exists at least one -invariant of \( N \) such that \( \langle \alpha \rangle \cap \{ p_1, p_2 \} = \{ p_i \} \), where \( i = 1 \) or 2.

**Proof:** The incidence matrices of \( N \) and \( N' \) have the following forms:

\[
V = \begin{pmatrix}
T_0 & T \\
V_0 & 0
\end{pmatrix}
\]

\[
V_1 = \begin{pmatrix}
T_1 & T_2 \\
V_{11} & V_{12} & V_{21} & V_{22}
\end{pmatrix}
\]

\[
V_2 = \begin{pmatrix}
T_1 & T_2 \\
V_{11} & V_{12} & V_{21} & V_{22}
\end{pmatrix}
\]

(1) Since is regular, is connected. It is obvious that is also connected.

(2) Since there exists a positive -invariant of \( N \) such that \( \alpha(p_1) = \alpha(p_2) \), let \( \alpha' = (\alpha(p_0) \alpha(p_1) \alpha(p_2)) \) and \( \alpha'V = \alpha(p_0)F_0 + \alpha(p_1)V_1 + \alpha(p_2)V_2 = 0 \). This means that is a positive -invariant of .

(3) Since is regular, has a positive -invariant. Let \( \beta \) be a positive -invariant of , then \( V \beta^T = 0 \). It is obvious that \( V^T \beta^T = 0 \). That is, \( \beta \) is also a -invariant of .

(4) Since \( p_1 \) and \( p_2 \) belong to the same cluster, \( |C(N')| = |C(N)| \). Suppose that there exists one -invariant \( \alpha \) such that \( \langle \alpha \rangle \cap \{ p_1, p_2 \} = \{ p_2 \} \). Then, the corresponding row of \( p_2 \) in \( V \) can be expressed as
a linear combination of the other rows of $P_0$. Hence, $\text{Rank}(V') = \text{Rank}(V)$. This means that $\text{Rank}(V') = \text{Rank}(V) = |\mathcal{C}(N)| - 1 = |\mathcal{C}(N')| - 1$. □

It is obvious from the proof of Theorem 2 that when $p_1$ and $p_2$ belong to different cluster, $N'$ must not be regular if $N$ is regular and Condition (3) holds.

3.3 Merging A Pair Of Transitions From Two Nets

COMPOSITION-BY-TRANSITION (composition via merging a pair of transitions from two different nets): Consider two disconnected ordinary nets $N_1 = (P_1, T_1 \cup \{t_1\}, F_1)$ and $N_2 = (P_2, T_2 \cup \{t_2\}, F_2)$, where $P_1 \cap P_2 = \emptyset$, $(T_1 \cup \{t_1\}) \cap (T_2 \cup \{t_2\}) = \emptyset$, $T_1 \cap \{t_1\} = \emptyset$, $T_2 \cap \{t_2\} = \emptyset$ and $F_1 \cap F_2 = \emptyset$. Let $N$ be composed from $N_1$ and $N_2$ by merging the pair of transitions $t_1$ and $t_2$. That is, $N = (P, T, F)$, where

$P = P_1 \cup P_2, T = T_1 \cup T_2 \cup \{t_1\}$ and $F = F_1 \cup F_2$.

Theorem 3: Let $N$, $N_1$ and $N_2$ be defined in COMPOSITION-BY-TRANSITION. Then, $N$ is regular if $N_1$ and $N_2$ are regular.

Proof: Similar to that of Theorem 1. □

3.4 Merging Two Non-neighboring Transitions In A Net

MERGE-N-TRANSITION (merging two transitions in a net): Let a net $N = (P, T_0 \cup \{t_1, t_2\}, F)$ be a net and satisfy the following conditions: $(T_0 \cup \{t_1\}) \cap (T_0 \cup \{t_2\}) = \emptyset$. Let $N' = (P, T_0 \cup \{t_1\}, F')$ be obtained from $N$ by merging the transitions $t_1$ and $t_2$ into a single place $t_{12}$, where $F' = F' = F \cup (\{(t_1, p) \mid (t_1, p) \in F \land (t_2, p) \in F\} \cup \{(p, t_1) \mid (p, t_1) \in F \land (p, t_2) \in F\} \cup \{(p, t_2) \mid (p, t_2) \in F \land (p, t_1) \in F\} \cup \{(t_1, p) \mid (t_1, p) \in F \land (t_2, p) \in F\}$, where $i = 1, 2$.

Note that since this paper just considers pure and ordinary nets, we add some conditions to $t_1$ and $t_2$, i.e., just considering non-neighboring transitions to be merged.

Theorem 4: Let $N$ and $N'$ be involved in MERGE-N-TRANSITION, where $N = (P, T_0 \cup \{t_1, t_2\}, F)$ and $N' = (P, T_0 \cup \{t_1\}, F')$. Then, $N'$ is regular if the following conditions hold:

1. $N$ is regular,
2. there exists a positive $T$-invariant $\beta$ such that $\beta(t_1) = \beta(t_2)$,
3. $t_1$ and $t_2$ belong to the same cluster, and
4. there exists at least one $T$-invariant $\beta$ such that $\beta(t_1) = \beta(t_2)$, where $i = 1$ or 2.

Proof: Similar to that of Theorem 2. □

4 PRESERVING LIVENESS AND BOUNDEDNESS

Desel and Esparza have shown that the regularity guarantees the existence of a live and bounded marking. This section will decide if a given initial marking $M_0$ ensures liveness and boundedness of $(N, M_0)$, where $N$ is the net obtained by applying the four compositional operations defined in Section 3.

Lemma 1 below characterizes liveness and boundedness of regular nets.

Lemma 1: (Desel and Esparza, 1995) Let $N$ be a regular net. Then, a marking $M$ of $N$ is live and bounded iff it marks all minimal siphons of $N$.

In order to check whether the liveness and boundedness of a marked regular net, it is important to know whether all siphons are marked after applying the four compositional operations. The following propositions state the relationship of siphons of nets of before and after transformation.

Proposition 1: Suppose that $N, N_1$ and $N_2$ are defined in COMPOSITION-BY-PLACE. Let $D$ be a siphon of $N$. Then, $D_1 = D \cap P_1$ is a siphon of $N_1$ if $p_{12} \notin D$, otherwise $D_1 = (D \cap P_1) \cup \{p_1\}$ is a siphon of $N_1$, where $i = 1$ and 2.

Proof: Since $D$ is a siphon of $N$, $D \subseteq D'$. Since $P_1 \cap P_2 = \emptyset$, $T_1 \cap T_2 = \emptyset$ and $D = (D \cap P_1) \cup (D \cap P_2)$, $D_1 \cup D_2$ if $p_{12} \notin D$, it is obvious that $D_1 \subseteq D_1'$ and $D_2 \subseteq D_2'$. This means that $D_1 = D \cap P_1$ is a siphon of $N_1$ if $p_{12} \notin D$. If $p_{12} \in D$, then $p_{12}$ in $N = (p_1$ in $N_1) \cup (p_2$ in $N_2)$ and $p_{12} \in N = (p_1$ in $N_1) \cup (p_2$ in $N_2)$. Since $D \subseteq D'$ and $T_1 \cap T_2 = \emptyset$, $(D \cap P_1) \cup \{p_1\} \subseteq ((D \cap P_1) \cup \{p_1\})$, this means that $(D \cap P_1) \cup \{p_1\}$ is a siphon of $N$, where $i = 1$ and 2. □

Proposition 2: Suppose that $N$ and $N'$ are defined in MERGE-N-PLACE. Let $D'$ be a siphon of $N'$. Then, $D'$ is a siphon of $N$ if $p_{12} \notin D'$, otherwise $D = D' \cup \{p_1, p_2\} - \{p_{12}\}$ is a siphon of $N$.

Proof: Since $D'$ is a siphon of $N$, $D' \subseteq D''$ in $N'$. It is obvious that $D' \subseteq D$ in $N$ if $p_{12} \notin D'$. This means that $D$ is a siphon of $N$ if $p_{12} \notin D''$, then $p_{12} \in N = (p_1 \in N_1) \cup (p_2 \in N_2)$ and $p_{12} \in N = (p_1 \in N_1) \cup (p_2 \in N_2)$. Since $D' \subseteq D''$ and $T_1 \cap T_2 = \emptyset$, $(D' \cup \{p_1, p_2\} - \{p_{12}\}) \subseteq (D' \cup \{p_1, p_2\} - \{p_{12}\})$, this means that $D' \cup \{p_1, p_2\} - \{p_{12}\}$ is a siphon of $N$. □

Proposition 3: Suppose that $N, N_1$ and $N_2$ are defined in COMPOSITION-BY-TRANSITION. Let $D$ be a siphon of $N$. Then, $D_1 = D \cap P_1$ is a siphon of $N_1$, where $i = 1$ or 2.

Proof: Since $P_1 \cap P_2 = \emptyset$, $T_1 \cap T_2 = \emptyset$ and $D = (D \cap P_1) \cup (D \cap P_2)$, $D_1 \cup D_2$, it is obvious that $D_1 \subseteq D_1'$ and $D_2 \subseteq D_2'$ because of $D \subseteq D'$. □
Proposition 4: Suppose that $N$ and $N'$ are defined in MERGE-N-TRANSITION. Let $D'$ be a siphon of $N'$. Then, $D'$ is a siphon of $N$ if $\{t_{12}\} \not\subseteq D'$.

Proof: Since $D'$ is a siphon of $N'$, $D' \subseteq D''$ in $N$. It is obvious that $D'' \subseteq D'$ in $N$ if $\{t_{12}\} \not\subseteq D'$, i.e., $D'$ is a siphon of $N$ if $\{t_{12}\} \not\subseteq D'$. □

Theorem 5: Suppose that $N$ and $N'$ are defined in COMPOSITION-BY-PLACE. Let $M_i$ be an initial marking of $N_i$ and $M$ be obtained from $M_i$ such that $M(p) = M_i(p)$ if $p \in P_i - \{p_1, p_2\}$ and $M(p_{12}) = \max\{M_i(p_1), M_2(p_2)\}$. Then, $(N, M)$ is live and bounded if $N_i$ is regular and $(N_i, M_i)$ is live and bounded, where $i = 1$ and 2.

Proof: Since $N_i$ and $N_2$ are regular, by Theorem 3.1, $N$ is regular. Since $(N_1, M_1)$ and $(N_2, M_2)$ are live and bounded, according to Lemma 1, all siphons of $N_1$ and $N_2$ are marked. Let $D$ be a siphon of $N$. By Proposition 1, if $p_{12} \notin D$, then $D_i = D \cap P_i$ is a siphon of $N_i$. Hence, $D_i$ is marked. If $p_{12} \in D$, then $(D \cap P) \cup \{p_i\}$ is a siphon of $N_i$ by Property 1 and thus $(D \cap P) \cup \{p_i\}$ is marked. In this case, if $p_i$ is marked for $i = 1$ or 2, then $p_{12}$ is marked, otherwise, $D \cap P$ is marked. This means that all siphons of $N$ are marked. Thus, $(M, N)$ is live and bounded according to Lemma 1. □

Theorem 6: Suppose that $N$ and $N'$ are defined in MERGE-N-PLACE, and $M$ is an initial marking of $N$. Let $M'$ be obtained from $M$ such that $M'(p) = M(p)$ if $p \in P - \{p_1, p_2\}$ and $M'(p_{12}) = \max\{M(p_1), M_2(p_2)\}$. Then, $(N', M')$ is live and bounded if the following conditions hold:

1. $(N, M)$ is regular,
2. $p_1$ and $p_2$ belong to the same cluster,
3. there exists a positive $P$-invariant of $N$ $\alpha$ such that $\alpha(p_1) = \alpha(p_2)$,
4. there exists at least one $P$-invariant $\alpha$ such that $\langle \alpha \rangle \cap \{p_1, p_2\} = \{p_i\}$ for $i = 1$ or 2, and
5. $(N, M)$ is live and bounded.

Proof: By Conditions (1)-(4) and Theorem 2, $N'$ is regular. Condition (5) implies that all siphons are marked according to Lemma 1. Consider any siphon $D'$ of $N'$. By Proposition 2, if $p_{12} \notin D'$, then $D = D'$ is a siphon of $N$. Hence, $D$ is marked. If $p_{12} \in D'$, then $(D \cap P) \cup \{p_1, p_2\}$ is a siphon of $N$ and thus is marked. This means that all siphons of $N'$ are marked. Thus, $(N, M)$ is live and bounded according to Lemma 1. □

The example below shows the application of some results obtained in Section 3 and this section.

Example 1: In Figure 1, both $(N_1, M_1)$ and $(N_2, M_2)$ are live and bounded marked graphs. Of course, $N_1$ and $N_2$ are regular. After applying COMPOSITION-BY-PLACE to them, $p_1$ and $p_2$ are merged into $r_1$ and $(N, M)$ shown in Figure 2 is obtained. By Theorem 1 and Theorem 5, $N$ is regular and $(N, M)$ is live and bounded. After applying MERGE-N-PLACE to $(N, M)$, $p_3$ and $p_4$ are merged into $r_2$ and $(N', M')$ shown in Figure 3 is obtained. Since, in $N_i$, $p_3$ and $p_4$ belong to the same cluster and $\{p_3, p_1, p_2\}$ is a $P$-component, by Theorem 2 and Theorem 6, $N'$ is also regular and $(N', M')$ is live and bounded.

Proof: Similar to the proof of Theorem 5. □

Figure 1: Two live and bounded Petri nets

Figure 2: Petri net $(N, M)$ obtained from Figure 1 by merging $p_1$ and $p_2$ into $r_1$.

Figure 3: Petri net $(N', M')$ obtained from Figure 2 by merging $p_3$ and $p_4$ into $r_2$. 

Theorem 7: Suppose that $N$ and $N_i$ are defined in COMPOSITION-BY-TRANSITION. Let $M_i$ be an initial marking of $N_i$ and $M$ be obtained from $M_i$ such that $M(p) = M_i(p)$ if $p \in P_i$. Then, $(N, M)$ is live and bounded if $N_i$ is regular and $(N_i, M_i)$ is live and bounded, where $i = 1$ and 2.

Proof: Similar to the proof of Theorem 5. □
Theorem 8: Suppose that $N$ and $N'$ are defined in MERGE-N-TRANSITION. Let $M$ be an initial marking of $N$ and $M'$ be obtained from $M$ such that $M'(p) = M(p)$ for any $p \in P$. Then, $(N', M')$ is live and bounded if the following conditions hold:

1. $N$ is regular,
2. $t_1$ and $t_2$ belong to the same cluster,
3. there exists a positive $T$-invariant $\beta$ such that $\beta(t_1) = \beta(t_2)$,
4. there exists at least one $T$-invariant $\beta$ such that $(\beta(t) \cap \{t_1, t_2\} = \{t_i\}$ for $i = 1$ or $2$, and
5. $(N, M)$ is live and bounded, and
6. all input places of $t_1$ and $t_2$ are marked or every minimal siphon $D'$ of $N'$ with $t_{12} \in D'$ is marked.

Proof: By Conditions (1)-(4) and Theorem 4, $N'$ is regular. Condition (5) implies that all siphons are marked according to Lemma 1. Consider any siphon $D'$ of $N'$. By Property 4, $D'$ is a siphon of $N$ if $\{t_{12}\} \not\subseteq D'$ and thus $D'$ is marked. By Condition (6), $D'$ is marked if $t_{12} \in D'$. This means that all siphons of $N'$ are marked. Hence, $(N, M)$ is live and bounded according to Lemma 1. □

5 CONCLUSION

This paper studied four compositional operations in terms of place and transition for pure and ordinary nets and showed that regularity, liveness and boundedness can be preserved automatically or under some simpler conditions. These compositional operations are quite natural. COMPOSITION-BY-PLACE and COMPOSITION-BY-TRANSITION are usually used to obtain more complex nets from some subnets. Liveness and boundedness preservations on the two operations for different subclasses of Petri nets have been studied under different conditions. Our results are based on the regularity preservation. MERGE-N-PLACE and MERGE-N-TRANSITION are two operations used in a net, a little similar to (Berthelot, 1987). Of course, these results that this paper are contributed are new and can accommodate the design of complex systems.

ACKNOWLEDGEMENTS

I would like to thank the anonymous referees for their helpful comments. The research was funded by the National Natural Science Foundation of China under Grants No. 60473007 and No. 60421001.

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