

ANALYSIS AND SYNTHESIS OF DIGITAL STRUCTURE BY MATRIX METHOD

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Abstract: This paper presents a general matrix algorithm for analysis and synthesis of digital filters. A useful method for computing the state-space matrix of a general digital network and a new technique for the design of digital filters are shown by means of examples. The method proposed in this paper allows the analysis of the digital filters and the construction of new equivalent structures of the canonic and non canonic digital filter forms. Equivalent filters with different structures can be found according to various matrix expansions. The procedure proposed in this paper is more efficient and economic than traditional methods because it permits to construct circuits with a minimum of shifting operations.

1 INTRODUCTION

The digital system presented in figure 1 is described by the following equations:

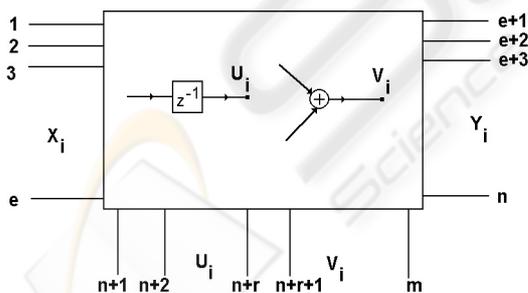


Figure 1: State-space structure with multiple inputs and outputs.

$$\begin{aligned} \mathbf{Y}(z) &= \mathbf{F}_{YX}\mathbf{X}(z) + \mathbf{F}_{YU}\mathbf{U}(z) + \mathbf{F}_{YV}\mathbf{V}(z) \\ \mathbf{U}(z) &= \mathbf{F}_{UX}\mathbf{X}(z) + \mathbf{F}_{UU}\mathbf{U}(z) + \mathbf{F}_{UV}\mathbf{V}(z) \\ \mathbf{V}(z) &= \mathbf{F}_{VX}\mathbf{X}(z) + \mathbf{F}_{VU}\mathbf{U}(z) + \mathbf{F}_{VV}\mathbf{V}(z) \end{aligned} \quad (1)$$

or in matrix form (2), (Luecker, 1976)

$$\mathbf{N}_s \times \begin{bmatrix} \mathbf{X}(z) \\ \mathbf{Y}(z) \\ \mathbf{U}(z) \\ \mathbf{V}(z) \end{bmatrix} = \mathbf{0}, \quad (2)$$

where \mathbf{N}_s in equation (2) is the signal flow matrix that represents the signal-flow graph of the digital system with multiple inputs and multiple outputs, $\mathbf{X}(z)$ is the vector of the input signals X_i , $\mathbf{Y}(z)$ is the vector of the output signals Y_i , $\mathbf{U}(z)$ is the vector of the signals U_i in the output of the delay elements and $\mathbf{V}(z)$ is a vector of the signals V_i in the output of the adders, see figure 1. \mathbf{N}_s can be obtained by expression (3), where \mathbf{F}_{YX} in the equation (3) is the transfer matrix output/input, $\mathbf{F}_{YX} = \mathbf{Y}(z)/\mathbf{X}(z)$ if $\mathbf{U}(z)=\mathbf{V}(z)=0$. In figure 2, signals U_3 and U_4 represent the outputs of the delay elements and signals V_5 and V_6 designate the outputs of the summers.

$$\mathbf{N}_s = \begin{bmatrix} \mathbf{F}_{(YX)} & -\mathbf{E} & \mathbf{F}_{(YU)} & \mathbf{F}_{(YV)} \\ \mathbf{F}_{(UX)} & \mathbf{0} & \mathbf{F}_{(UU)} - \mathbf{E} & \mathbf{F}_{(UV)} \\ \mathbf{F}_{(VX)} & \mathbf{0} & \mathbf{F}_{(VU)} & \mathbf{F}_{(VV)} - \mathbf{E} \end{bmatrix} \quad (3)$$

If we reduce the signals avoiding the outputs of the adders V_i in expression (2), we obtain

$$\mathbf{N}_e \times \begin{bmatrix} \mathbf{X}(z) \\ \mathbf{Y}(z) \\ \mathbf{U}(z) \end{bmatrix} = \mathbf{0}, \quad (4)$$

where \mathbf{N}_e in equation (4) is a flow-state matrix and the matrices \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{D} in the matrix equation (5) are the state matrices of the digital system.

$$\mathbf{N}_e = \left[\begin{array}{c|c|c} \mathbf{D} & -\mathbf{E} & \mathbf{C} \\ \hline z^{-1}\mathbf{B} & \mathbf{0} & z^{-1}\mathbf{A} - \mathbf{E} \end{array} \right] \quad (5)$$

In the flow-state matrix, the matrices \mathbf{E} and $\mathbf{0}$ are identity and zero matrices respectively. If we reduce the matrix equation (4), not taking into account the vector of the signals U_i , we get the expression

$$\mathbf{N}_t^{(2)} \times \begin{bmatrix} \mathbf{X}(z) \\ \mathbf{Y}(z) \end{bmatrix} = \mathbf{0}, \quad (6)$$

where the transfer matrix $\mathbf{N}_t^{(2)}$ can be defined by (7)

$$\mathbf{N}_t^{(2)} = \left[\mathbf{D} + \mathbf{C} \times (z\mathbf{E} - \mathbf{A})^{-1} \mathbf{B}; \quad -\mathbf{E} \right] \quad (7)$$

The element $n_{21}^{(2)}$ of the transfer matrix $\mathbf{N}_t^{(2)}$ is the transfer function $H(z)$ of the digital network.

$$n_{21}^{(2)} = H(z) = \mathbf{D} + \mathbf{C} \times (z\mathbf{E} - \mathbf{A})^{-1} \mathbf{B} \quad (8)$$

Using the inverse z transform, the impulse response of the circuit results (Luecker, 1976)

$$h(n) = \begin{cases} \mathbf{D} & \text{for } l = 0 \\ \mathbf{C}\mathbf{A}^{l-1}\mathbf{B} & \text{for } l > 0 \end{cases} \quad (9)$$

where $l=1,2,3 \dots$

2 ANALYSIS OF THE SECOND ORDER STATE-SPACE DIGITAL FILTER

As an example, we will determine the transfer function of a state-space second order digital filter, see figure 2. To calculate the signal-flow matrix of the digital filter, previously it is necessary to mark the input, output and state nodes (Pšenička and Herrera, 1997), (Pšenička and Ugalde, 1999). The input node is denoted by number 1 and the output node by number 2. The nodes 3 and 4 are assigned to the outputs of the delay elements. Finally, the nodes 5 and 6 are placed on the output of the adders. The system equations (10) for each node can be obtained from figure 2.

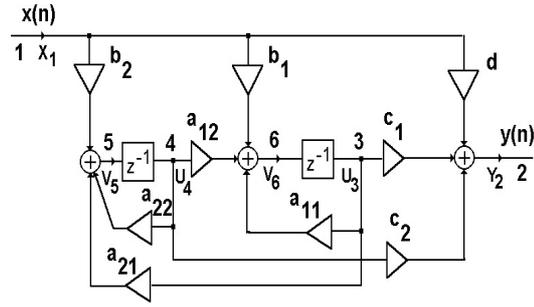


Figure 2: State-space digital filter of second order.

$$\begin{aligned} 2 : Y_2 &= X_1 d + U_3 c_1 + U_4 c_2 \\ 3 : U_3 &= V_6 z^{-1} \\ 4 : U_4 &= V_5 z^{-1} \\ 5 : V_5 &= X_1 b_2 + U_3 a_{21} + U_4 a_{22} \\ 6 : V_6 &= X_1 b_1 + U_3 a_{11} + U_4 a_{12} \end{aligned} \quad (10)$$

The equations (10) can be written by the matrix equation to generate the signal flow matrix $\mathbf{N}_s^{(6)}$. The first row of the matrix (11) is indexed by number 2, see also expression (10).

$$\mathbf{N}_s^{(6)} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} d & -1 & c_1 & c_2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & z^{-1} \\ 0 & 0 & 0 & -1 & z^{-1} & 0 \\ b_2 & 0 & a_{21} & a_{22} & -1 & 0 \\ b_1 & 0 & a_{11} & a_{12} & 0 & -1 \end{bmatrix} \end{matrix} \quad (11)$$

From the signal-flow matrix (11) we can observe, that the main diagonal contain -1's and in the second column all the elements are zeros except the first one.

The signal-flow matrix can be formed directly without writing node equations (10). For example, due to the delay element placed between the nodes 6 and 3, see figure 2, the matrix element $n_{36}^{(6)}$ of the matrix $\mathbf{N}_s^{(6)}$ acquires the value z^{-1} . The multiplier a_{21} located between the nodes 3 and 5 is represented in the matrix $\mathbf{N}_s^{(6)}$ by the element $n_{53}^{(6)}$ equal to the constant a_{21} . Similarly, we can obtain all of the elements in the signal-flow matrix without writing the nodal equations. The matrix $\mathbf{N}_s^{(6)}$ can be reduced to the matrix $\mathbf{N}^{(5)}$ equation (13) according to the expression

$$n_{ij}^{(k-1)} = \frac{n_{ij}^{(k)} n_{kk}^{(k)} - n_{ik}^{(k)} n_{kj}^{(k)}}{n_{kk}^{(k)}}, \quad (12)$$

where i represents the number of the row, j the number of the column and k the degree of the matrix.

Following the rule of reduction (12), we obtain the matrix $\mathbf{N}^{(5)}$ and the state-flow matrix $\mathbf{N}_e^{(4)}$

$$\mathbf{N}^{(5)} = \begin{bmatrix} d & -1 & c_1 & c_2 & 0 \\ z^{-1}b_1 & 0 & -1 + a_{11}z^{-1} & a_{12}z^{-1} & 0 \\ 0 & 0 & 0 & -1 & z^{-1} \\ b_2 & 0 & a_{21} & a_{22} & -1 \end{bmatrix} \quad (13)$$

$$\mathbf{N}_e^{(4)} = \left[\begin{array}{c|cc} d & -1 & \\ \hline z^{-1}b_1 & 0 & -1 + a_{11}z^{-1} \\ z^{-1}b_2 & 0 & a_{21}z^{-1} \end{array} \middle| \begin{array}{cc} c_1 & c_2 \\ a_{12}z^{-1} & -1 + a_{22}z^{-1} \end{array} \right] \quad (14)$$

If we compare the expressions (14) and (5), we obtain the state matrices **A**, **B**, **C**, and **D** of the state-space digital filter.

$$\begin{aligned} \mathbf{D} &= d & \mathbf{C} &= [c_1 \quad c_2] \\ \mathbf{B} &= \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} & \mathbf{A} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{aligned} \quad (15)$$

3 DESIGN OF THE THIRD ORDER STATE-SPACE STRUCTURE

In this example we are going to obtain the third order state-space structure. The state matrices of the third order state-space filter have the general form (16), (Psenicka et al., 1998).

$$\begin{aligned} \mathbf{D} &= d & \mathbf{C} &= [c_1 \quad c_2 \quad c_3] \\ \mathbf{B} &= \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} & \mathbf{A} &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{aligned} \quad (16)$$

Substituting (16) in (5) it is obtained the state-flow matrix (17)

$$\mathbf{N}_e^{(5)} = \left[\begin{array}{c|ccccc} d & -1 & c_1 & c_2 & c_3 \\ \hline z^{-1}b_1 & 0 & -1 + a_{11}z^{-1} & a_{12}z^{-1} & a_{13}z^{-1} \\ z^{-1}b_2 & 0 & a_{21}z^{-1} & -1 + a_{22}z^{-1} & a_{23}z^{-1} \\ z^{-1}b_3 & 0 & a_{31}z^{-1} & a_{32}z^{-1} & -1 + a_{33}z^{-1} \end{array} \right] \quad (17)$$

To expand the state-flow matrix (17) which contains five columns and four rows in the matrix with six columns and five rows (19), we use the equation (18).

The equation (18) is obtained from equation (12) for $n_{kk}^{(k)} = -1$.

$$n_{ij}^{(k)} = n_{ij}^{(k-1)} - n_{ik}^{(k)} n_{kj}^{(k)} \quad (18)$$

If we choose the elements of the new matrix $n_{26}^{(6)} = n_{46}^{(6)} = n_{56}^{(6)} = 0$, then the first, third and fourth rows in the new matrix $\mathbf{N}^{(6)}$ remain unchanged (19).

$$\mathbf{N}^{(6)} = \begin{bmatrix} d & -1 & c_1 & c_2 & c_3 & 0 \\ n_{31}^{(6)} & n_{32}^{(6)} & n_{33}^{(6)} & n_{34}^{(6)} & n_{35}^{(6)} & n_{36}^{(6)} \\ z^{-1}b_2 & 0 & a_{21}z^{-1} & -1 + a_{22}z^{-1} & a_{23}z^{-1} & 0 \\ z^{-1}b_3 & 0 & a_{31}z^{-1} & a_{32}z^{-1} & 1 - a_{33}z^{-1} & 0 \\ n_{61}^{(6)} & n_{62}^{(6)} & n_{63}^{(6)} & n_{64}^{(6)} & n_{65}^{(6)} & n_{66}^{(6)} \end{bmatrix} \quad (19)$$

The elements of the matrix (19), $n_{61}^{(6)}$, $n_{62}^{(6)}$, $n_{63}^{(6)}$, $n_{64}^{(6)}$, $n_{65}^{(6)}$, $n_{66}^{(6)}$ and $n_{36}^{(6)}$ can be chosen and the remaining elements $n_{31}^{(6)}$, $n_{32}^{(6)}$, $n_{33}^{(6)}$, $n_{34}^{(6)}$ and $n_{35}^{(6)}$ are obtained by means of the equation (18). The elements in the last row and columns of the matrix $\mathbf{N}^{(6)}$ must be chosen, in order to obtain in the second row of the matrix $\mathbf{N}^{(6)}$ plenty of zeros. It is suitable to choose the element $n_{36}^{(6)} = z^{-1}$, because all elements in the second row of the matrix $\mathbf{N}_e^{(5)}$ contain z^{-1} . But it is possible select the element $n_{36}^{(6)}$ in a different way, as we shall see in section 4.2. If we choose

$$\begin{aligned} n_{26}^{(6)} &= 0 & n_{46}^{(6)} &= 0 & n_{56}^{(6)} &= 0 & n_{62}^{(6)} &= 0 \\ n_{66}^{(6)} &= -1 & n_{65}^{(6)} &= a_{13} & n_{64}^{(6)} &= a_{12} & n_{63}^{(6)} &= a_{11} \\ n_{36}^{(6)} &= z^{-1} & n_{61}^{(6)} &= b_1 & & & & \end{aligned}$$

then we get by equation 18 the elements of the new matrix in the form

$$\begin{aligned} n_{31}^{(6)} &= n_{31}^{(5)} - n_{36}^{(6)} n_{61}^{(6)} = z^{-1}b_1 - z^{-1}b_1 & &= 0 \\ n_{32}^{(6)} &= n_{32}^{(5)} - n_{36}^{(6)} n_{62}^{(6)} = 0 - z^{-1}0 & &= 0 \\ n_{33}^{(6)} &= n_{33}^{(5)} - n_{36}^{(6)} n_{63}^{(6)} = -1 + a_{11}z^{-1} - a_{11}z^{-1} & &= -1 \\ n_{34}^{(6)} &= n_{34}^{(5)} - n_{36}^{(6)} n_{64}^{(6)} = z^{-1}a_{12} - z^{-1}a_{12} & &= 0 \\ n_{35}^{(6)} &= n_{35}^{(5)} - n_{36}^{(6)} n_{65}^{(6)} = z^{-1}a_{13} - z^{-1}a_{13} & &= 0 \end{aligned}$$

and we obtain the matrix $\mathbf{N}^{(6)}$

$$\mathbf{N}^{(6)} = \begin{bmatrix} d & -1 & c_1 & c_2 & c_3 & 0 \\ 0 & 0 & -1 & 0 & 0 & z^{-1} \\ z^{-1}b_2 & 0 & a_{21}z^{-1} & -1 + a_{22}z^{-1} & a_{23}z^{-1} & 0 \\ z^{-1}b_3 & 0 & a_{31}z^{-1} & a_{32}z^{-1} & -1 + a_{33}z^{-1} & 0 \\ b_1 & 0 & a_{11} & a_{12} & a_{13} & -1 \end{bmatrix} \quad (20)$$

Similarly, we can obtain the matrices $\mathbf{N}^{(7)}$ and $\mathbf{N}^{(8)}$. After a very simple calculation, we can get the matrix (21) and the signal-flow matrix (22). For example it is advantageous to choose the element $n_{47}^{(7)} = z^{-1}$, in the matrix $\mathbf{N}^{(7)}$, because each element in row 3 of the matrix $\mathbf{N}^{(6)}$ contains z^{-1} . In case the matrix element $n_{71}^{(7)}$ equal to b_2 is chosen, the element $n_{41}^{(7)}$ is equal to zero, marked by $|0|$. For example in order to obtain in the matrix $\mathbf{N}^{(7)}$ $n_{41}^{(7)} = 0$ if $n_{41}^{(6)} = b_2 \cdot z^{-1}$ it is necessary to choose $n_{47}^{(7)} = z^{-1}$ and $n_{71}^{(7)} = b_2$ or viceversa. So to obtain in the matrix $\mathbf{N}^{(8)}$ $n_{55}^{(8)} = -1$ if $n_{55}^{(7)} = -1 + a_{33} \cdot z^{-1}$ it is necessary to choose $n_{58}^{(8)} = z^{-1}$ and $n_{85}^{(8)} = a_{33}$ or the contrary. The same procedure can be applied to equation (21) in order to get equation (22).

$$\mathbf{N}^{(7)} =$$

$$\begin{bmatrix} d & -1 & c_1 & c_2 & c_3 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & z^{-1} & 0 \\ |0| & 0 & 0 & -1 & 0 & 0 & |z^{-1}| \\ z^{-1}b_3 & 0 & a_{31}z^{-1} & a_{32}z^{-1} & -1 + a_{33}z^{-1} & 0 & 0 \\ b_1 & 0 & a_{11} & a_{12} & a_{13} & -1 & 0 \\ |b_2| & 0 & a_{21} & a_{22} & a_{23} & 0 & -1 \end{bmatrix} \quad (21)$$

$$\mathbf{N}^{(8)} = \begin{bmatrix} d & -1 & c_1 & c_2 & c_3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & z^{-1} & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & z^{-1} \\ b_1 & 0 & a_{11} & a_{12} & a_{13} & -1 & 0 & 0 \\ b_2 & 0 & a_{21} & a_{22} & a_{23} & 0 & -1 & 0 \\ b_3 & 0 & a_{31} & a_{32} & a_{33} & 0 & 0 & -1 \end{bmatrix} \quad (22)$$

The digital structure that corresponds to the signal flow matrix $\mathbf{N}^{(8)}$ is presented in figure 3.

The second canonic form of the state-space digital filter can be obtained from the structure presented in figure 3. Changing the adders to nodes, the nodes to adders, the input to output and the directions of the multipliers, the second canonic form of the state-space filter can be obtained. If other values are chosen for elements in the last row and the last column in matrices (19), (20) and (21) we can obtain other equivalent structure.

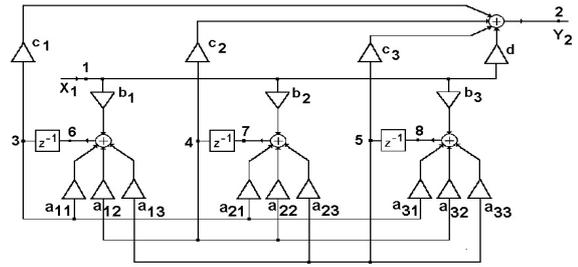


Figure 3: Third order state-space filter.

4 EXAMPLES

4.1 Design of the filter from the state space matrices

In the first example we shall demonstrate how to derive the structures of the state-space filter without multipliers if the state-space matrices \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{D} are known (23).

$$\mathbf{D} = 0.25 \quad \mathbf{C} = [0.25 \quad 0.5] \quad (23)$$

$$\mathbf{B} = \begin{bmatrix} 0.75 \\ 0.75 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} -0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

With the assistance of equation (5) we obtain the state-space matrix $\mathbf{N}_e^{(4)}$ in the form

$$\mathbf{N}_e^{(4)} = \begin{bmatrix} 2^{-2} & -1 & 2^{-2} & 2^{-1} \\ (2^{-2} + 2^{-1})z^{-1} & 0 & -1 + 2^{-1}z^{-1} & 2^{-1}z^{-1} \\ (2^{-2} + 2^{-1})z^{-1} & 0 & 2^{-1}z^{-1} & -1 + 2^{-1}z^{-1} \end{bmatrix} \quad (24)$$

Provided that we choose elements $n_{ij}^{(5)}$ in the matrix $\mathbf{N}^{(5)}$ in this way $n_{25}^{(5)} = 0$, $n_{35}^{(5)} = z^{-1}$, $n_{45}^{(5)} = 0$ and $n_{55}^{(5)} = -1$ we obtain the equation (25) from the equation (24). In the new matrix $\mathbf{N}^{(5)}$ the elements in the last row and last column can be chosen. The rest elements of the matrix $\mathbf{N}^{(5)}$ must be calculated by using the equation (18).

$$\mathbf{N}^{(5)} = \begin{bmatrix} 2^{-2} & -1 & 2^{-2} & 2^{-1} & 0 \\ 0 & 0 & -1 & 0 & z^{-1} \\ (2^{-2} + 2^{-1})z^{-1} & 0 & 2^{-1}z^{-1} & -1 + 2^{-1}z^{-1} & 0 \\ 2^{-2} + 2^{-1} & 0 & -2^{-1} & 2^{-1} & -1 \end{bmatrix} \quad (25)$$

In case that we choose elements $n_{ij}^{(6)}$ in the matrix $\mathbf{N}^{(6)}$ in this manner $n_{26}^{(6)} = 0$, $n_{36}^{(6)} = 0$, $n_{46}^{(6)} = z^{-1}$, $n_{56}^{(6)} = 0$ and $n_{66}^{(6)} = -1$ we obtain the equation (26) from the equation (25) .

$$\mathbf{N}^{(6)} = \begin{bmatrix} 2^{-2} & -1 & 2^{-2} & 2^{-1} & 0 & 0 \\ 0 & 0 & -1 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & -1 & 0 & z^{-1} \\ 2^{-2} + 2^{-1} & 0 & -2^{-1} & 2^{-1} & -1 & 0 \\ 2^{-2} + 2^{-1} & 0 & 2^{-1} & 2^{-1} & 0 & -1 \end{bmatrix} \quad (26)$$

If we choose elements $n_{ij}^{(7)}$ in the matrix $\mathbf{N}^{(7)}$ in this way $n_{27}^{(7)} = 0$, $n_{37}^{(7)} = 0$, $n_{47}^{(7)} = 0$, $n_{57}^{(7)} = z^{-1}$, $n_{67}^{(7)} = 0$ and $n_{77}^{(7)} = -1$ we obtain the equation (27) from the equation (26) .

$$\mathbf{N}^{(7)} = \begin{bmatrix} 2^{-2} & -1 & 2^{-2} & 2^{-1} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & z^{-1} & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & z^{-1} & 0 \\ 2^{-2} & 0 & 0 & 0 & -1 & 0 & 2^{-1} \\ 2^{-2} + 2^{-1} & 0 & 2^{-1} & 2^{-1} & 0 & -1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 & -1 \end{bmatrix} \quad (27)$$

Provided that we choose elements $n_{ij}^{(8)}$ in the matrix $\mathbf{N}^{(8)}$ in this manner $n_{28}^{(8)} = 0$, $n_{38}^{(8)} = 0$, $n_{48}^{(8)} = 0$, $n_{58}^{(8)} = 0$, $n_{68}^{(8)} = 2^{-1}$, $n_{78}^{(8)} = 0$ and $n_{88}^{(8)} = -1$ we obtain the equation (28) from the equation (27). The equation (28) is the signal flow-matrix and from this matrix the circuit can be sketched. The structure that correspond to the signal flow matrix $\mathbf{N}^{(8)}$ is presented in figure 5.

$$\mathbf{N}_s^{(8)} = \begin{bmatrix} 2^{-2} & -1 & 2^{-2} & 2^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & z^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & z^{-1} & 0 & 0 \\ 2^{-2} & 0 & 0 & 0 & -1 & 0 & 2^{-1} & 0 \\ 2^{-2} & 0 & 0 & 0 & 0 & -1 & 0 & 2^{-1} \\ 1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \quad (28)$$

In figure 4 there is a classical structure of the state-space filter of the second order that has 11 shift operations. In figure 5 the proposed equivalent state-space

structure of the second order is presented with only 7 shift operations. It can be easily verified that both structures have the same impulse response.

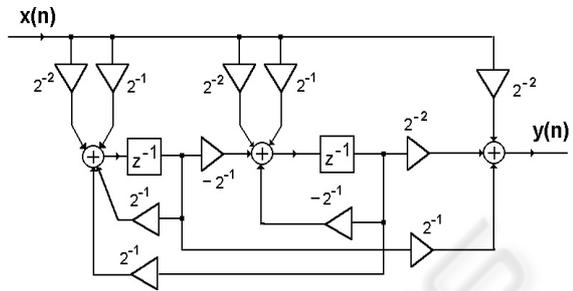


Figure 4: Classical state-space structure of the second order.

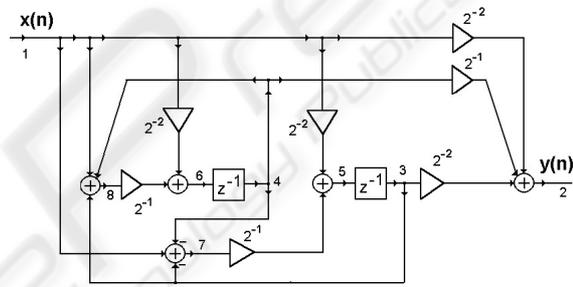


Figure 5: State-space structure without multipliers.

In the second example, we shall calculate the elliptic low-pass state space filter of the third order, $n=3$, with attenuation in the band-pass $a_{max} = 2$ dB, corner frequency $f_1 = 0.6$ and attenuation in the band-stop $a_{min} = 15$ dB. By means of MATLAB command $[a,b,c,d] = \text{ellip}(3,2,15,0.6)$ we obtain the state-space matrices for the elliptic low-pass filter with the values:

$$\mathbf{C} = [0.1887 \quad 0.0360 \quad 0.0680]$$

$$\mathbf{D} = 0.3673$$

$$\mathbf{B}^T = [2.1724 \quad 0.9698 \quad 1.3201]$$

$$\mathbf{A} = \begin{bmatrix} 0.1160 & 0.0000 & 0.0000 \\ 0.4982 & -0.3513 & -0.8830 \\ 0.6782 & 0.8830 & -0.2019 \end{bmatrix}$$

With the following equations that realize low-pass state-space filter of the 3rd order we can obtain the impulse response $y_n(i)$ in the frequency domain. The

equations for calculating $y_n(i)$, n_6 , n_7 , n_8 etc. were derived from figure 3.

```

a11=0.1160;a12=0.0000;a13=0.0000;
a21=0.4982;a22=-0.3513;a23=-0.8830;
a31=0.6782;a32=0.8830;a33=-0.2019;
b1=2.1724;b2=0.9698;b3=1.3201;
c1=0.1887;c2=0.0360;c3=0.0680;d=0.3673;
n3=0;n4=0;n5=0;
xn=1;
for i=1:1:200
    yn(i)=xn*d+n3*c1+n4*c2+n5*c3;
    n6=xn*b1+n3*a11+n4*a12+n5*a13;
    n7=xn*b2+n3*a21+n4*a22+n5*a23;
    n8=xn*b3+n3*a31+n4*a32+n5*a33;
    n3=n6;n4=n7;n5=n8;xn=0;
end
[h,w]=freqz(yn,1,200);
plot(w,20*log10(abs(h)))
    
```

In the third example we shall calculate the high-pass elliptic filter with the specification $n=3$, $a_{max} = 2$ dB, $a_{min} = 15$ dB and corner frequency $f_{-1} = 0.4$. By means of MATLAB commands `[a,b,c,d]=ellip(3,2,15,0.4,'high')` we obtain the state-space matrices for the elliptic high-pass filter in the form

$$C = \begin{bmatrix} -0.2597 & -0.0495 & -0.0936 \end{bmatrix}$$

$$D = 0.3673$$

$$B^T = \begin{bmatrix} 1.5783 & 0.7046 & 0.9591 \end{bmatrix}$$

$$A = \begin{bmatrix} -0.1160 & 0.0000 & 0.0000 \\ -0.4982 & 0.3513 & 0.8830 \\ -0.6782 & -0.8830 & 0.2019 \end{bmatrix}$$

For the analysis of the high-pass filter we have used the same equation as for the low-pass, but the constants a_{ij} , b_i and c_i in the program must be changed.

In the fourth example we shall realize the elliptic band-pass state-space filter for the lower and upper corner frequencies 0.4 and 0.6 respectively, and $a_{min} = 15$ dB. Using MATLAB commands we get the state matrices **a**, **b**, **c** and **d**.

```

[a,b,c,d]=ellip(3,3,15,[0.4,0.6])
a =
-0.1374    0    0    0.8626    0    0
 0.2458   -0.1229   -0.2791    0.2458    0.8771   -0.2791
 0.0782    0.2791   -0.0888    0.0782    0.2791    0.9112
-0.8626    0    0    0.1374    0    0
-0.2458   -0.8771    0.2791   -0.2458    0.1229    0.2791
-0.0782   -0.2791   -0.9112   -0.0782   -0.2791    0.0888
b =
 0.7927
 0.2259
 0.0719
-0.7927
-0.2259
-0.0719
c =
 0.1125   -0.0022    0.0420    0.1125   -0.0022    0.0420
d =
 0.1034
    
```

The general structure for the state-space filter of the arbitrary order can be derived from the structure in figure 3. To analyze the structure by MATLAB in the figure 3, the following algorithm must be used. The attenuation of the band-pass state-space filter is presented in figure 6.

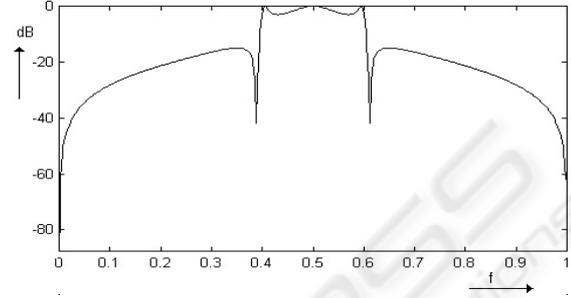


Figure 6: Attenuation of the band-pass state-space Caer Filter.

```

A11=-0.137;A12=0.000;A13=0.000;A14=0.862;A15=0.000;
A16=0.000;A21=0.245;A22=-0.122;A23=-0.279;A24=0.245;
A25=0.877;A26=-0.279;A31=0.078;A32=0.279;A33=-0.088;
A34=0.078;A35=0.279;A36=0.911;A41=-0.862;A42=0.000;
A43=0.000;A44=0.137;A45=0.000;A46=0.000;A51=-0.245;
A52=-0.877;A53=0.279;A54=-0.245;A55=0.122;A56=0.279;
A61=-0.078;A62=-0.279;A63=-0.911;A64=-0.078;
A65=-0.279;A66=0.088;B1=0.792;B2=0.225;B3=0.071;
B4=-0.792;B5=-0.225;B6=-0.071;C1=0.112;C2=-0.002;
C3=0.042;C4=0.112;C5=-0.002;C6=0.042;D=0.1034;
N3=0;N4=0;N5=0;N6=0;N7=0;N8=0;XN=1;
for i=1:1:500
    YN(i)=D*XN+N3*C1+N4*C2+N5*C3+N6*C4+N7*C5+N8*C6;
    N9 =B1*XN+N3*A11+N4*A12+N5*A13+N6*A14+N7*A15+N8*A16;
    N10=B2*XN+N3*A21+N4*A22+N5*A23+N6*A24+N7*A25+N8*A26;
    N11=B3*XN+N3*A31+N4*A32+N5*A33+N6*A34+N7*A35+N8*A36;
    N12=B4*XN+N3*A41+N4*A42+N5*A43+N6*A44+N7*A45+N8*A46;
    N13=B5*XN+N3*A51+N4*A52+N5*A53+N6*A54+N7*A55+N8*A56;
    N14=B6*XN+N3*A61+N4*A62+N5*A63+N6*A64+N7*A65+N8*A66;
    N3=N9;N4=N10;N5=N11;N6=N12;N7=N13;N8=N14;XN=0;
end
[h,w]=freqz(YN,1,500);
plot(w,20*log10(abs(h)))
    
```

4.2 Design of the filter from the transfer function by matrix method

In this part we shall derive the structure of the digital filter without multipliers that has the transfer function (29)

$$H(z) = \frac{P(z)}{Q(z)} = \frac{0.3437 - 0.2890z^{-1} + 0.4296z^{-2}}{1 + 0.0625z^{-1} - 0.4218z^{-2}} \quad (29)$$

The constants of the transfer function (29) can be decomposed into the following expression

$$\begin{aligned} 0.34375 &= 2^{-2} + 2^{-4} + 2^{-5} \\ 0.2890 &= 2^{-2} + 2^{-5} + 2^{-7} \\ 0.4296 &= 2^{-2} + 2^{-3} + 2^{-5} + 2^{-6} + 2^{-7} \\ 0.0625 &= 2^{-4} \\ 0.4218 &= 2^{-2} + 2^{-3} + 2^{-5} + 2^{-6} \end{aligned}$$

First of all the matrix $\mathbf{N}^{(3)}$ in the form (30) must be written

$$\mathbf{N}^{(3)} = \begin{bmatrix} 0 & -1 & 1 \\ P(z) & 0 & -Q(z) \end{bmatrix} \quad (30)$$

$\mathbf{N}^{(3)} =$

$$\left[\begin{array}{c|c|c} 0 & -1 & 1 \\ \hline 2^{-2} + 2^{-4} + 2^{-5} & 0 & -1 - z^{-1}2^{-4} \\ -z^{-1}(2^{-2} + 2^{-5} + 2^{-7}) & & +z^{-2}(2^{-2} + 2^{-3} + 2^{-5} + 2^{-6}) \\ \hline 1 - z^{-1} + z^{-2} & 0 & z^{-2} \\ \hline 1 - z^{-1} + z^{-2} & 0 & z^{-2} \end{array} \right] \quad (31)$$

In this case we shall expand transfer matrix $N^{(3)}$ equation (31), so the signal U_i must be added. If we choose in the matrix $\mathbf{N}^{(4)}$ the element $n_{24}^{(4)} = 0$ the elements in the first row remain unchanged. In case that $n_{34}^{(4)} = 2^{-5}$ and $n_{41}^{(4)} = 1 - z^{-1} - z^{-2}$ are chosen then element $n_{31}^{(4)}$ in the matrix $\mathbf{N}^{(4)}$ acquire the value $2^{-2} + 2^{-4} - z^{-1}(2^{-2} + 2^{-7}) + z^{-2}(2^{-2} + 2^{-3} + 2^{-6} + 2^{-7})$. Similarly if the element $n_{43}^{(4)} = z^{-2}$ will be chosen then the element $n_{33}^{(4)}$ in the matrix $\mathbf{N}^{(4)}$ acquire the value $-1 - z^{-1}2^{-4} + z^{-2}(2^{-2} + 2^{-3} + 2^{-6})$. The element $n_{44}^{(4)}$ must be equal -1.

$\mathbf{N}^{(4)} =$

$$\left[\begin{array}{c|c|c|c} 0 & -1 & 1 & 0 \\ \hline 2^{-2} + 2^{-4} - & 0 & -1 - z^{-1}2^{-4} + & 2^{-5} \\ z^{-1}(2^{-2} + 2^{-7}) & & z^{-2}(2^{-2} + 2^{-3} + 2^{-6}) & \\ +z^{-2}(2^{-2} + 2^{-3} & & & \\ +2^{-6} + 2^{-7}) & & & \\ \hline 1 - z^{-1} + z^{-2} & 0 & z^{-2} & -1 \end{array} \right] \quad (32)$$

In the new matrix $\mathbf{N}^{(5)}$ the elements in the last row and last column can be chosen. Provided that we choose elements $n_{ij}^{(5)}$ in the matrix $\mathbf{N}^{(5)}$ in this way $n_{25}^{(5)} = 0$, $n_{35}^{(5)} = 2^{-2}$, $n_{45}^{(5)} = 0$, $n_{55}^{(5)} = -1$, we obtain $n_{51}^{(5)} = 1 - z^{-1} + z^{-2}$, $n_{52}^{(5)} = 0$, $n_{53}^{(5)} = z^{-2}$ and $n_{54}^{(5)} = 0$ and we get the equation (33) from the equation (32).

$\mathbf{N}^{(5)} =$

$$\left[\begin{array}{c|c|c|c|c} 0 & -1 & 1 & 0 & 0 \\ \hline 2^{-4} - z^{-1}(2^{-7}) & 0 & -1 - z^{-1}2^{-4} & 2^{-5} & 2^{-2} \\ +z^{-2}(2^{-3} + 2^{-6} & & +z^{-2}(2^{-3} + 2^{-6}) & & \\ +2^{-7}) & & & & \\ \hline 1 - z^{-1} + z^{-2} & 0 & z^{-2} & -1 & 0 \\ \hline 1 - z^{-1} + z^{-2} & 0 & z^{-2} & 0 & -1 \end{array} \right] \quad (33)$$

In the new matrix $\mathbf{N}^{(6)}$ the elements in the last row and last column can be chosen again. In case that we choose elements $n_{ij}^{(6)}$ in the matrix $\mathbf{N}^{(6)}$ in this manner $n_{26}^{(6)} = 0$, $n_{36}^{(6)} = 2^{-3}$, $n_{46}^{(6)} = 0$, $n_{56}^{(6)} = 0$, $n_{66}^{(6)} = -1$, we get $n_{61}^{(6)} = z^{-2}$, $n_{62}^{(6)} = 0$, $n_{63}^{(6)} = z^{-2}$, $n_{64}^{(6)} = 0$, $n_{65}^{(6)} = 0$ and we obtain the equation (34) from the equation (33).

$\mathbf{N}^{(6)} =$

$$\left[\begin{array}{c|c|c|c|c|c} 0 & -1 & 1 & 0 & 0 & 0 \\ \hline 2^{-4} - z^{-1}2^{-7} & 0 & -1 - z^{-1}2^{-4} & 2^{-5} & 2^{-2} & 2^{-3} \\ +z^{-2}(2^{-6} + 2^{-7}) & & +z^{-2}2^{-6} & & & \\ \hline 1 - z^{-1} + z^{-2} & 0 & z^{-2} & -1 & 0 & 0 \\ \hline 1 - z^{-1} + z^{-2} & 0 & z^{-2} & 0 & -1 & 0 \\ \hline z^{-2} & 0 & z^{-2} & 0 & 0 & -1 \end{array} \right] \quad (34)$$

In case that we choose elements $n_{ij}^{(7)}$ in the matrix $\mathbf{N}^{(7)}$ in this manner $n_{27}^{(7)} = 0$, $n_{37}^{(7)} = 2^{-6}$, $n_{47}^{(7)} = 0$, $n_{57}^{(7)} = 0$, $n_{67}^{(7)} = 0$, $n_{77}^{(7)} = -1$, we get $n_{71}^{(7)} = z^{-2}$, $n_{72}^{(7)} = 0$, $n_{73}^{(7)} = z^{-2}$, $n_{74}^{(7)} = 0$, $n_{75}^{(7)} = 0$, $n_{76}^{(7)} = 0$ and we obtain the equation (35) from the equation (34).

$\mathbf{N}^{(7)} =$

$$\left[\begin{array}{c|c|c|c|c|c|c} 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ \hline 2^{-4} - z^{-1}2^{-7} & 0 & -1 - z^{-1}2^{-4} & 2^{-5} & 2^{-2} & 2^{-3} & 2^{-6} \\ +z^{-2}2^{-7} & & & & & & \\ \hline 1 - z^{-1} & 0 & z^{-2} & -1 & 0 & 0 & 0 \\ +z^{-2} & & & & & & \\ \hline 1 - z^{-1} & 0 & z^{-2} & 0 & -1 & 0 & 0 \\ +z^{-2} & & & & & & \\ \hline z^{-2} & 0 & z^{-2} & 0 & 0 & -1 & 0 \\ \hline z^{-2} & 0 & z^{-2} & 0 & 0 & 0 & -1 \end{array} \right] \quad (35)$$

Following with this procedure we obtain the matrix $\mathbf{N}_s^{(14)}$. The matrix $\mathbf{N}_s^{(14)}$ is the signal-flow matrix and from this matrix the circuit can be drawn. The structure that correspond to the signal-flow matrix (36) is presented in the figure 7 with minimum shift operations. In the figure 8 A) is shown the equivalent filter with 16 shift operations and the figure 8 B) displays the filter with 5 multipliers. The circuits in the figure 7 and 8 have the same impulse response.

